Vote Trading in Public Elections (updated version)

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Abstract

During the 2000 U.S. Presidential race an apparently new idea, called vote trading, was introduced to help one of the two major-party candidates (Gore) win. The idea was, through an Internet mechanism, to induce voters who supported a minor-party candidate (Nader) to vote for Gore in states where this would help Gore and to induce an equal number of voters who supported Gore to vote for Nader in states where this would not hurt Gore. Thus Nader would receive the same number of popular votes as he would have received without the trading (providing an incentive for Nader voters to participate). Vote trading was implemented at a number of Web sites in 2000 (and again in 2004); it illustrates how information technology can be used to exploit the electoral college system; and it has the potential to alter the outcome of Presidential elections. In this paper, we formalize this idea, present several variations, and present an optimal way for Web sites to implement it (so as to best help the major-party candidate get elected) in both deterministic and stochastic settings.

Keywords: voting strategy, integer programming, knapsack problem.

JEL Codes: C61, C44, D72

1 Introduction

In the 2000 U.S. Presidential race there were two major-party candidates, Al Gore and George W. Bush, and one significant minor-party candidate, Ralph Nader. In the months before the election, Gore and Bush were close in the polls and Nader was a distant third. On September 23, 2000, an interesting, and apparently new, voting strategy was reported by James Ridgeway in the Village Voice (see Ridgeway (2000)). (He attributed the idea to "Nader supporters.") The strategy, subsequently dubbed vote trading or vote swapping, was designed to increase Gore's chances of winning the election. The strategy was presented with more detail by Jamin Raskin (a law professor at American University) in an article appearing October 25, 2000 in Slate magazine (see Raskin (2000)).

Vote trading (as described in Raskin (2000)) worked as follows.[1] The first step was to identify two types of states: "swing" states where Gore and Bush were neck and neck (like Florida) and "lost" states where Bush had a big lead (like Texas). The idea was then, through an Internet mechanism, to pair up Nader supporters (whose second choice was Gore) in swing states with Gore supporters in lost states and have them trade votes; that is, the Nader supporters would vote for Gore and the Gore supporters would vote for Nader. Thus Gore would have a better chance of winning the electoral votes in the swing states and hence the election. One incentive for Nader supporters to participate was that their second choice candidate would be helped. Another is that Nader would receive the same number of popular votes nationally as he would have received without the trading. This was important since a minimum number (5%) of the popular votes was needed to obtain federal funding in the next election.

Vote trading was, apparently, first implemented on October 1, 2000 at the Web site voteexchange.org. On October 25, 2000 the idea was implemented again at NaderTrader.org. Soon after this, several more Web sites were set up to implement this strategy: e.g., VoteSwap2000.com, VoteExchange2000.com, WinWinCampaign.org, and VoteExchange.com.[2] Each site invited certain types of voters to sign up. Whenever a compatible pair of voters had signed up, they were immediately instructed via e-mail to switch their vote in the election. It appears that all sites implementing the strategy were set up by individuals, independent of any political party. (Brief histories of vote trading appear in Randazza (2001), Davis (2002), and Raskin (2003).) According to a symposium

[1] Vote trading exploits the electoral college system. Recall: The candidate winning the most popular votes in a state wins all the electoral votes of that state (except in two states); the candidate winning a majority of the electoral votes nationwide (i.e., at least 270 out of 538 total electoral votes) is elected President. The number of electoral votes in a state is equal to the number of Representatives plus Senators in the state (hence it is approximately proportional to the population of the state).

[2] The legality of vote trading quickly became an issue (see Derfner (2000)). Because no money is exchanged in vote trading, it appears to be legal, although the issue has not been officially resolved (see Randazza (2001) and Raskin (2003), where the authors argue for its legality). In January 2001, the Washington College of Law at American University held a symposium on vote trading, including discussions of its legality (see Symposium (2001)).

announcement (in which several of the vote trading Web site creators participated; see Symposium (2001)), around 36,000 voters participated in vote trading by signing up at one of the Web sites. Recall that Bush received only 537 more votes than Gore in the contested state of Florida and whichever candidate won this state won the election. Thus a bit more vote trading in Florida could have altered the outcome of the election.

Vote trading was again adopted in the 2004 U.S. Presidential election. This time it was implemented at just one Web site: VotePair.org. That site reported that 21,992 people signed up at their site wishing to trade votes and 2659 pairs were arranged.

A variation of vote trading was also adopted in 2001 and 2005 in the U.K. at several Web sites to help left-of-center candidates win seats in Parliament (see Ledbetter (2001), Biever (2005), and Section 7 of this paper for further discussion; also see www.tacticalvoter.net, a Web site for U.K. vote trading where it is claimed that vote trading may have affected the outcome in two elections in 2001 because the number of vote trading pairs exceded the margins of victory).

An important consideration in an analysis of vote trading is the phenomenon of cheating. For example, a Bush supporter in 2004 who wanted to dilute the effect of vote trading could sign up at a Web site claiming to be a Nader supporter in a swing state. Such a person, if selected for a pair, would vote for Bush and potentially displace a true Nader supporter who was not selected and would have legitimately switched their vote in that state if instructed. Another possibility is a Nader supporter signing up in a swing state who, if selected for a pair, would still vote for Nader and hope their partner, a Kerry supporter in a Bush state, would also vote for Nader.

Vote trading organizers in the 2000 and 2004 Presidential elections were well aware of the possibilities for cheating and took steps to reduce it. One important step was to put the pairs of selected traders into e-mail contact with each other. The idea was that a dialogue between traders could build trust and reduce cheating. If a selected person did not trust their partner, they could request another. Another standard Web security measure was implemented in 2004 to reduce automated or computerized sign ups: The sign up process required the typing of a word that was presented in a computer-unreadable format.

In the future, other methods may arise for reducing the occurance of cheating. For example, when voters can vote by mail (used for all state elections in Oregon since 1998) or vote on the Internet, then pairs of vote traders could "prove" to one another that they have voted a certain way (e.g., by using organized, independent observers or software).

In general, the phenomenon of vote trading illustrates how modern computer technology in the hands of individuals can have surprising effects on the mechanics of election procedures. In particular, vote trading has the potential to alter the outcomes of elections (Florida in 2000 and the U.K. in 2001).

The main purposes of this paper are to present the following:

- a formal model of the idea of vote trading;
- new ways for Web sites to implement vote trading in situations like the U.S. elections;
- the best way for Web sites to implement vote trading (so as to get a specified candidate elected), under certain assumptions;
- new situations in which variations of vote trading can be used; and
- a technique for analyzing the performance of vote trading schemes.

An additional hope for the paper is that increasing our understanding of vote trading and its variations may add an extra dimension to discussions concerning the design of election procedures (for example, vote trading may serve as a new component in arguments against using the electoral college system in the U.S.).

The general approach of the paper is to first present and study a deterministic model (in Sections 2-4) of vote trading, where the uncertainties concerning how people are going to vote (from polling) and the level of cheating are not addressed. We then add these uncertainties to the model (in Section 5) and propose two methods for implementing vote trading, one of which uses the deterministic approach as a subroutine. We next discuss the paper in more detail.

The paper begins, in Section 2, by presenting a formal (deterministic) model for vote trading. This allows us to precisely describe a variety of ways in which the basic idea can be implemented in practice. The basic parameters of the model are as follows:

- the number of Web sites;
- the types of voters who are invited to sign up at these Web sites;
- at what time signed-up voters are instructed to switch or not.

In Section 3 we present a technique for comparing different choices of these parameters and show that one choice dominates all others in a fairly strong sense (it is the best at finding vote trades that win the election for a specified candidate). This "optimal strategy" has the following properties: it has only one Web site; it allows trades between more types of voters than the original strategies in Ridgeway (2000) and Raskin (2000); and it delays its choice of trades until shortly before the election. Thus our optimal strategy differs from the original implementation of vote trading in the U.S. in 2000 on all three basic parameters. This optimal strategy also requires the solution of what we call the "vote trading problem," which is the determination of which signed-up voters to instruct to switch. (It is defined in Section 2.)

In Section 4 we explore the relationship between the vote trading problem and the field of integer programming. First, we show that the vote trading problem is polynomially equivalent to an integer program. Then, using this integer program, we show that the vote trading problem is polynomially equivalent to a knapsack problem. Thus we show that, in polynomial time, any vote trading problem can be reformated as a knapsack problem and, conversely, that any knapsack problem can be reformulated as a vote trading problem. Thus we see that the vote trading problem is NP-hard; however, the wealth of techniques for solving integer programs and knapsack problems (e.g., see Martello and Toth (1990) and Nemhauser and Wolsey (1988)) can be used to find solutions in practice.

In Section 5 we consider the notion of uncertainty. In particular, we consider the uncertainties related to predicting how people will vote (through polling) and the amount of cheating. We address the problem of finding solutions to the vote trading problem that maximize the *probability* of a candidate winning. We show how this problem can be solved by reducing it to the solution of a number of deterministic versions of the problem. A key parameter of this model is an estimate of the amount of cheating. Thus the probability of vote trading winning the election for various estimates of cheating can be explored. In particular, this approach can be used to help decide whether or not to implement vote trading, by computing "best case" estimates of the potential effectiveness of vote trading (i.e., when the amount of cheating is low).

In section 6 we briefly discuss how the approach in this paper can also be used by cheaters to address the problem of minimizing the probability of vote trading being effective.

Finally, Section 7 contains a brief discussion of some of our model assumptions, some new variations and applications of the idea of vote trading, and some open problems.

We next relate a little history of this paper. In early August of 2004, a draft was sent to Jamin Raskin (mentioned above; also a legal advisor and press contact for VotePair.org), Carnet Williams (originator of WinWinCampaign.org in 2000 and a coordinator for VotePair.org), and Alan Porter (originator of VoteExchange2000.com and a backer of VotePair.org). No vote trading Web sites for the 2004 election had been announced or launched at that time. When vote trading for the 2004 election first appeared on the Web (on Sept. 20, 2004), it was implemented at a single site: VotePair.org. Furthermore, instructions to switch were delayed until mid-October with the following reason given at the site:

"When we match you with a voting partner, we optimize your pairing to maximize its effectiveness and safety: we want to make sure that your participation wins electoral votes for Kerry without losing any. Accumulating a large number of participants to pair all at once helps us to do this."

The method of choosing the pairs was not described at VotePair.org. Thus, two key suggestions in this paper were implemented.

Let us end this section by putting the notion of vote trading into a broader context. A primary motivation for the organizers of vote trading in the U.S. has been a belief that the U.S. electoral system is flawed (e.g., a candidate can win a majority of the popular votes and lose the election; see VotePair.org). Vote trading attempts to address such issues with a method that can be implemented immediately. However, other election methods have been proposed that could address these issues more directly (although changing the U.S. electoral system is likely to be difficult since it favors smaller states and three-fourths of the states must ratify an amendment to the U.S. constitution). A method favored by the organizers of VotePair.org (see the Web site) is called instant runoff voting (or the Hare system), which is a special case of single-transferable voting. This method allows individual voters to rank *all* the candidates according to their preferences. Variants are used, for example, in major elections in both Australia and the Republic of Ireland. Other well-known alternative methods are the Borda count and approval voting. See Mueller (2003) for a thorough introduction to these and other methods. See Saari (2001) for a less technical introduction, where vote trading is briefly mentioned.

Another type of vote trading (which is distinct from the topic of this paper) occurs between members of government bodies when voting for various issues. Sometimes called logrolling, this topic has been widely studied. See Mueller (2003) for an introduction; some key papers are those by Bentley (1907), Buchanan and Tullock (1962), and Riker and Brams (1967).

This paper appears to be the first mathematical study of the properties of vote trading in public elections. The mathematical techniques used are from the fields of operations research/statistics and there is some work in the voting literature that is similar in this regard. For example, a paper by Koford (1982) uses mathematical modeling to study logrolling. Also, three papers by Bartholdi, Tovey, and Trick, (1989a),(1989b), and (1992), and a paper by Bartholdi and Orlin (1991) use the idea of NP-completeness to study a variety of voting schemes.

2 The Models

In this section we introduce our notation, terminology, assumptions, and the details of our deterministic models and strategies. Ultimately, we introduce several "vote trading systems," which are well-defined implementions of the general idea of vote trading. This begs the question, "Which system is best"? This is the topic of the next section. We also introduce, in this section, the "vote trading problem," which is a problem that must be solved in implementing the best strategy. We analyze this problem in Section 4.

The following list introduces some of our notation and assumptions. Undefined details in A4 are given later in this section. Relaxations of assumptions A1, A4, and A7 are discussed in Section 7.

Basic notation and assumptions:

A1: There are three candidates, denoted A, B, C, running for President (using the electoral college system described in footnote 1).

A2: We let X denote the set of states. For each state $i \in X$, we let e_i denote the number of electoral votes in i.

A3: For each state $i \in X$, we have numbers a_i, b_i, c_i, which are the numbers of voters who prefer candidates A, B, C, respectively. (Hence we assume we know with certainty how voters will vote for these candidates; in Section 5 we relax this assumption.) For simplicity, we assume, for each state $i \in X$, that $a_i > c_i$ and $b_i > c_i$ (which were clearly true in the 2000 and 2004 U.S. Presidential elections, where C was Nader).

A4: At time T_s one or more Web sites are set up. Each site makes the following announcement:

"Voters who prefer B (and satisfy property P_B) and voters who prefer C (and satisfy property P_C) are invited to sign up at this site.

"Voters who sign up must provide their e-mail address, candidate preference, and state of residence. This information is put onto one of two lists: L_B or L_C for voters who prefer B and C, respectively. Periodically, we will choose an equal number of voters from each list and immediately instruct them to switch their vote, from B to C or from C to B, respectively, when they vote in the election. These voters' names will then be removed from the lists. We will operate until time T_e.

"Our objective is to get B elected. We will instruct voters to switch so that the number of electoral votes for B is maximally increased."

A5: Voters sign up at no more than one Web site; truthfully reveal their e-mail address, preferred candidate, and their state of residence when signing up; and switch their vote, if instructed. (The issue of cheating is addressed in Sections 5 and 6.)

A6: Web sites do not share any information from their lists with each other.

In practice, T_s might be chosen to be two months before the election and T_e might be chosen to be one week before the election.

Let us next consider the incentives for the voters to sign up. Observe that each Web site satisfies two key properties:

Prop. 1: The objective is to get B elected; voters are instructed to switch so that the number of electoral votes for B is greater than or equal to the number of electoral votes for B if no voters were instructed to switch.

Prop. 2: C receives the same number of votes, nationwide, both with and without the site.

For a voter who prefers B, Property 1 is an incentive to sign up at a Web site. For a voter whose first choice is C, Properties 1 and 2 are an incentive to sign up if the following conditions hold:

- the voter's second choice is B;
- C has essentially no chance of winning the election;
- C needs to win a certain percentage of the nationwide popular vote total in order to get future funding and C is close to having this percentage.

In the 2000 Presidential race, we had this situation where A was Bush, B was Gore, and C was Nader (who wanted to win 5% of the nationwide vote). In the 2004 race, we nearly had this situation where A was Bush, B was Kerry, and C was Nader (the support for Nader appeared to be lower than in 2000).

Below we list three questions that must be answered to make the Web sites described in A4 well defined:

Ques. 1: What are properties P_B and P_C?

Ques. 2: What is meant by "periodically"?

Ques. 3: How are the voters chosen from the lists?

Answer to question 1: To define the possibilities for P_B and P_C, we begin with the following definitions:

1. A state $i \in X$ is called a *lost state* if B is not ahead in i and no amount of vote switching from C to B in i could change this. That is, $a_i \geq b_i + c_i$.

2. A state $i \in X$ is called a *won state* if B is ahead in i. That is, $b_i > a_i$.

3. A state $i \in X$ is called a *swing state* if B is not ahead in i, but there are enough votes for C in i so that switching votes from C to B could give B a win. That is, $a_i > b_i$ and $a_i < b_i + c_i$.

Observe that the lost states, won states, and swing states form a partition of X (given our assumption A3). There is no advantage in inviting voters who prefer C in lost or won states to sign up, hence Web sites consider inviting only the following types of voters:

Type 1: Voters in lost states who prefer B.

Type 2: Voters in won states who prefer B.

Type 3: Voters in swing states who prefer B.

Type 4: Voters in swing states who prefer C.

Observe that if a Web site instructs only type 1 and type 4 voters to switch, then the number of electoral votes for B cannot decrease. Hence, since each Web site has the objective of getting B elected, we assume P_C contains all voters of type 4. We focus on two special cases of P_B. We say a Web site is using a *restricted strategy* if P_B is all type 1 voters, and no others, or is all type 1 and type 2 voters, and no others; and a site is using an *unrestricted strategy* if P_B is all type 1, 2, and 3 voters.

To simplify our analysis, we make the following assumption (see discussion in Section 7).

A7: Each Web site uses either a restricted or unrestricted strategy.

The article Raskin (2000) suggested that Web sites should invite voters of types 1 and 4, and all that were set up in 2000 and 2004 did so. The article Ridgeway (2000) suggested that Web sites should invite voters of types 1, 2, and 4 (and at least one Web site did invite voters of type 2 (see Williams (2004))). It appears that no Web site in 2000 or 2004 invited voters of type 3 to sign up (we see in the proof of Proposition 6 that inviting these types of voters can improve the performance of vote trading).

Answer to question 2: We begin with the following definition:

Definition 1 *Consider a Web site with lists L_B and L_C at some time t between T_s and T_e. Two voters $l_B \in L_B$ and $l_C \in L_C$ are called* tradeable *if they satisify the following two conditions:*

1. *They are from different states;*

2. *Instructing them to switch would not reduce the number of electoral votes for B at time t (where we consider previously instructed switches as being made).*

Remark 2 *A type 1 and type 4 voter are always tradeable.*

We consider two extreme strategies. A Web site is said to employ a *real-time strategy* if it chooses and instructs voters to switch as soon as it has a tradeable pair of voters signed up. A Web site is said to employ a *delayed-time strategy* if it chooses and instructs voters to switch only once, at time T_e. It appears that most if not all Web sites set up in 2000 used a real-time strategy, whereas the single Web site set up in 2004 used a delayed-time strategy.

Answer to question 3:

Observe that a Web site will typically have many ways of choosing a set of voters to instruct when such choices are made. We assume each Web site employs a well-defined *algorithm* that chooses these sets. The algorithm takes as input two lists, L_B and L_C, of voters and outputs two equal-sized subsets of the two lists (possibly empty) who are instructed to switch; the algorithm may or may not employ randomization. We assume such an algorithm obeys the following restrictions when it is run:

For sites using a real-time strategy:

- Select from L_B and L_C a tradeable pair that, if switched, would maximally increase the number of electoral votes for B (in combination with previously instructed switches; the increase may be zero). Break ties arbitrarily.

For sites using a delayed-time strategy:

- Select subsets $L'_B \subseteq L_B$ and $L'_C \subseteq L_C$ such that $|L'_B| = |L'_C|$ and instructing these voters to switch would maximize the number of electoral votes for B.

It is not difficult to see that, for a site using a real-time strategy, the algorithm can be implemented in polynomial (in fact linear) time (by the definition of a real-time strategy, the algorithm is run immediately after a new person signs up, hence this person will be in the tradeable pair).

A site using a delayed-time strategy has one key problem it must solve (finding the subsets defined above), at time T_e, which we call the *vote trading problem.* The number of equal-sized pairs of subsets to be considered in solving this problem is, in general, exponential in $|L_B \cup L_C|$; so it is not immediately clear if this problem can be solved in polynomial time (although it can clearly be solved by enumeration; note that, in order to solve this problem when there are multiple Web sites, a site must observe the strategies employed by the other sites). Analyzing the vote trading problem is the focus of Section 4.

A collection of Web sites, each satisfying the conditions detailed above, is called a *vote trading system.* A vote trading system is called *centralized* if it has exactly one Web site; otherwise it is called *decentralized.*

To summarize, we have defined the following types of vote trading systems: Centralized or decentralized where each site uses either a restricted or unrestricted strategy together with either a real-time or delayed-time strategy. A vote trading system is independent of the particular set of states X and the distribution of voter types within these states.

The subject of the next section is to determine which of these systems is "best." (There are a few more types of vote trading systems that can be considered if assumptions A4 and/or A7 are relaxed. We address this issue in Section 7.)

3 Comparing Vote-Trading Systems

Our objective in this section is to compare the performance of the vote trading systems defined in the previous section. We begin by defining the means we use for making comparisons. We then show that a centralized system, whose site uses unrestricted and delayed-time strategies, is "as good as" any other system. This much is intuitively obvious and is easy to show. But we also are able to show a stronger result: such a system is "strictly better" than any other system. The proof of this is essentially a sequence of examples demonstating that a centralized system, whose site uses unrestricted and delayed-time strategies can perform strictly better than any alternative system.

Suppose we have a set of states X (with numbers a_i, b_i, c_i for each $i \in X$) and a set of voters V from these states (of types 1-4) who are willing to sign up in a vote trading system. Let S be an arbitrary vote trading system. An *instance* $I(X, V, S)$ is a choice by each voter in V of a site in S that will accept their type (if there is such a site) and a time to sign up at that site between T_s and T_e. Observe that a vote trading system S takes as input an instance $I(X, V, S)$ and produces a pair of sets of voters, say U_B and U_C, who are instructed to switch, from B to C and from C to B, respectively.

A vote trading system S is said to *dominate* a system S' if for every set of states X, every set of voters V, and every pair of instances, $I(X, V, S)$ and $I(X, V, S')$, the number of electoral votes for B under S is greater than or equal to the number of electoral votes for B under S'. S is said to *strictly dominate* S' if S dominates S' and there exists a set of states X, a set of voters V, and a pair of instances $I(X, V, S)$ and $I(X, V, S')$ such that the number of electoral votes for B under S is strictly greater than the number of electoral votes for B under S'.

Let V be the set of voters and, for any vote trading system S, let $V_B(S)$ and $V_C(S)$ denote the subsets of V who sign up at some site in S and prefer B and C, respectively. We let S^* denote the vote trading system that is centralized and the Web site uses unrestricted and delayed-time strategies. We first show that S^* dominates all other vote trading systems and then, through a series of propositions, that S^* strictly dominates all other systems.

Proposition 3 *The vote trading system S^* dominates all other vote trading systems.*

Proof. Let X and V be arbitrary sets of states and voters, respectively; let S be an arbitrary vote trading system; and let $I(X, V, S)$ be an arbitrary instance. Because the site in S^* is unrestricted, we have

$V_B(S) \subseteq V_B(S^*)$ and $V_C(S) = V_C(S^*)$. Therefore, since S^* searches all possible sets of voters to switch, it will consider all possible sets that can be output by S. Hence the number of electoral votes for B under S is at least the number of electoral votes for B under S^*. ■

Proposition 4 *S^* strictly dominates all decentralized vote trading systems.*

Proof. Let S be a decentralized vote trading system and let $W1$ and $W2$ be two distinct Web sites in S. Let $\{S1, S2\} \subseteq X$, where $S1$ is a lost state and $S2$ is a swing state. Let V_B be a set of 1000 B voters in $S1$ and let V_C be a set of 1000 C voters in $S2$. Let $V = V_B \cup V_C$. Suppose $S2$ has 100 electoral votes, and B needs 1000 more votes to win in $S2$. Consider an instance $I(X, V, S)$ where all the voters in V_B sign up at $W1$ and all the voters in V_C sign up at $W2$. Then neither $W1$ nor $W2$ can instruct any voters to switch (whether they use restricted or unrestricted, real-time or delayed-time strategies). However, under every instance $I(X, V, S^*)$, B wins $S2$ and gains 100 electoral votes, since the single Web site can instruct all the voters in V to switch. ■

Proposition 5 *S^* strictly dominates all centralized vote trading systems where the site uses a real-time strategy.*

Proof. Consider a centralized vote trading system S that uses a real-time strategy. Suppose $\{S1, S2, L1\} \subseteq X$, where $S1$ and $S2$ are swing states with 100 and 70 electoral votes, respectively, and $L1$ is a lost state. Suppose, in each of $S1$ and $S2$, B needs 1000 more votes to win the state. Consider an instance $I(X, V, S)$ with the following properties: 1000 type 1 voters from $L1$ sign up at the site; next, 1000 type 4 voters from $S2$ sign up at the site; finally, 1000 type 4 voters from $S1$ sign up and no further voters sign up. Then the algorithm for S instructs the voters from $L1$ and $S2$ to switch (using Assumption A7), thus gaining 70 electoral votes for B. However, B gains 100 electoral votes under every instance $I(X, V, S^*)$ since S^* can instruct 1000 voters in each of $S1$ and $L1$ to switch. ■

Proposition 6 *S^* strictly dominates all centralized vote trading systems where the site uses restricted and delayed-time strategies.*

Proof. Let S be a centralized vote trading system where the site uses restricted and delayed-time strategies. Suppose $\{S1, S2\} \subseteq X$, where both states are swing states. Suppose $S1$ has 100 electoral votes and 1000 voters in $S1$ must switch from C to B in order for B to win $S1$.

Suppose 1000 C voters in $S1$ are willing to sign up and 1000 B voters in $S2$ are willing to sign up. Let V denote these 2000 voters. Then B gains no votes in any instance $I(X, V, S)$, since the voters in $S2$ cannot sign up. However, in any instance $I(X, V, S^*)$, all 2000 voters are instructed to switch and B gains 100 electoral votes. ■

Thus we have shown that S^* strictly dominates all the other vote trading systems introduced in Section 2.

4 IP Formulation, Knapsack Equivalence, and NP-hardness

In this section we show that the vote trading problem for S^* is polynomially equivalent to an integer programming problem. We then show that this IP can be reformulated, in polynomial time, as a knapsack problem, hence the techniques used to solve knapsack problems can be used to solve the vote trading problem. We then show the converse: that an arbitrary knapsack problem can be reformulated, in polynomial time, as a vote trading problem for some S^*. Actually we show a somewhat stronger result, that a knapsack problem can be reformulated as a vote trading problem for a vote trading system using a restricted strategy where an additional condition, called "proportionality" (defined below), holds. Thus we conclude that the vote trading problem for S^* is NP-hard. We begin with a few definitions.

Suppose we have the vote trading system S^*. For each state $i \in X$, let V_i^B and V_i^C be the sets of voters who have signed up at S^* by time T_e, are from state i, and prefer candidates B and C, respectively. Let $V^B = \cup_{i \in X} V_i^B$ and $V^C = \cup_{i \in X} V_i^C$.

Then the vote trading problem for S^* is to find subsets $\overline{V}^B \subseteq V^B$ and $\overline{V}^C \subseteq V^C$, such that $\left|\overline{V}^B\right| = \left|\overline{V}^C\right|$ and if we instruct the voters in these subsets to switch then the number of electoral votes for B is maximized.

Consider the following properties that a state $i \in X$ may satisfy at time T_e, before any votes are switched:

P1: B is ahead in i and if all voters in V_i^B switch (and no voters in V_i^C switch), then B remains ahead in i.

P2: B is ahead in i and if all voters in V_i^B switch (and no voters in V_i^C switch), then A becomes ahead, or A and B tie for first place in i.

P3: A is ahead in i and if all voters in V_i^C switch (and no voters in V_i^B switch), then B becomes ahead in i.

P4: A is ahead in i and if all voters in V_i^C switch (and no voters in V_i^B switch), then A remains ahead, or A and B tie for first place in i.

By assumption A3, either A or B is ahead in i before any switching occurs. Let us further make the following reasonable assumption:

A8: Switching all the voters in V_i^B cannot win the state for C. (This assumption is not necessary for getting the type of results in this section; however, the results become a bit more complex without it.)

Then (under assumption A8) it follows that every state satisfies exactly one of these properties. Let P^1, P^2, P^3, P^4 denote the sets of states that satisfy properties P1,...,P4, respectively.

We next show that the vote trading problem is polynomially equivalent to an integer program. What we mean by this is that an optimal solution to the vote trading problem can be transformed into an optimal solution to the IP in polynomial time, and vice versa.

Our IP makes use of the following variables (which constitute an *incidence vector* for the won states, after switching):

$$x_i = \begin{cases} 1 \text{ if candidate } B \text{ wins in state } i \text{ after switching} \\ 0 \text{ if candidate } A \text{ wins (or ties with } B\text{) in state } i \text{ after switching} \end{cases}$$

Proposition 7 *The vote trading problem is polynomially equivalent to the following integer program:*

$$\max \sum_{all\ i} e_i x_i$$

$$\sum_{i \in P^3} (a_i - b_i + 1)\, x_i \le \sum_{i \in P^1 \cup P^4} \left|V_i^B\right| + \sum_{i \in P^2} (b_i - a_i - 1)\, x_i + \sum_{i \in P^2 \cup P^3} \left|V_i^B\right| (1 - x_i) \tag{IP}$$

$$x_i \in \{0, 1\} \text{ for each } i \in X.$$

Proof. First, we show that for any feasible solution to the vote trading problem there exists a feasible solution to the above IP such that the two solutions have the same objective value; second, we show the converse of this.

Let $\overline{V}^B$ and $\overline{V}^C$ be a feasible solution to a vote trading problem for S^*. Construct x^*, the associated incidence vector of won states. Then x^* is a feasible solution for the constraint in the above IP since the value of the LHS of IP, call it $LHS(x^*)$, is a lower bound on the number of votes that must be switched from C to B and the value of the RHS, call it $RHS(x^*)$, is an upper bound on the number of votes that can be switched from B to C. That is, $LHS(x^*) \leq \left|\overline{V}^C\right| = \left|\overline{V}^B\right| \leq RHS(x^*)$. Clearly, the solutions to the two problems have the same objective values.

Conversely, let x^* be a feasible solution for the above IP. We construct a feasible solution to the vote trading problem as follows:

Let $LHS(x^*)$ be defined as above. Arbitrarily choose $LHS(x^*)$ voters from $\overline{V}^C$ in those states in P^3 for which $x_i^* = 1$ and instruct them to switch from C to B; and choose $LHS(x^*)$ voters from $\overline{V}^B$ in the remaining states and instruct them to switch from B to C (with the exception that no more than $b_i - a_i - 1$ voters are chosen from the states in P^2 for which $x_i^* = 1$; and no voters from the states in P^2 and P^3, where $x_i^* = 1$, are chosen). These choices are possible since x^* is feasible for the IP; furthermore, these choices yield a feasible solution for the vote trading problem that wins the states indicated by x^*. Again, the solutions to the two problems have the same objective values. ■

The above proof shows how to transform an optimal solution to the IP into an optimal solution to the vote trading problem. The next result shows that the IP is polynomially equivalent to a knapsack problem. Consider the following maximization version of the knapsack problem, stated as an integer program (see Garey and Johnson (1979)), which is a well-known NP-hard problem:

Given: For $i = 1, \ldots, n$, *sizes* $s_i \in Z^+$ and *values* $v_i \in Z^+$; a number $K \in Z^+$.

Solve:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} v_i x_i \\ & \sum_{i=1}^{n} s_i x_i \leq K \\ & x_i \in \{0,1\} \text{ for } i = 1, \ldots n \end{aligned}$$

Proposition 8 *The vote trading problem is polynomially equivalent to the following knapsack problem:*

$$\max \sum_{all\ i} e_i x_i$$

$$\sum_{i \in P^2 \cup P^3} \left(a_i - b_i + \left|V_i^B\right| + 1\right) x_i \leq \sum_{i \in X} \left|V_i^B\right| \qquad \text{(K)}$$

$$x_i \in \{0,1\} \text{ for each } i \in X.$$

Proof. The result follows immediately from Proposition IP and the observation that the inequalities in IP and K are equivalent. ■

Thus we have shown that any vote trading problem can be reformulated, in polynomial time, as a knapsack problem. We next consider the converse of this: that any knapsack problem can be reformulated, in polynomial time, as a vote trading problem (which satisfies proportionality, defined below); thus the vote trading problem is NP-hard. We begin with a definition.

Definition 9 *A vote trading system satisfies proportionality if there is some value d such that, in each state $i \in X$, $\left\lfloor \frac{a_i+b_i+c_i}{e_i} \right\rfloor = d$.*

"Proportionality" captures the idea of the number of electoral votes being proportional to the population (or the number of voters) in each state, which is essentially the case in the Presidential election system.

Proposition 10 *The vote trading problem for a centralized vote trading system using delayed-time and restricted strategies is NP-hard, when proportionality holds. (Hence this problem is NP-hard when proportionality does not hold.)*

Proof. Consider an arbitrary instance of the knapsack problem. We show how it can be polynomially reduced to a vote trading problem for a centralized vote trading system using delayed-time and restricted strategies, where proportionality holds. In particular, the vote trading system contains one lost state and several swing states.

For each $i = 1, \ldots, n$, we associate a state S_i, whose parameters we now define. Arbitrarily choose $a_i, b_i \in Z^+$ so that $a_i - b_i + 1 = |V_i^C| = c_i := s_i$. Also, set $e_i := v_i$ and $|V_i^B| = 0$.

Next add a state S_{n+1} and set $|V_{n+1}^B| := K$. Arbitrarily choose $a_{n+1}, b_{n+1} \in Z^+$ so that $b_{n+1} = a_{n+1} - 1 > 0$, $b_{n+1} \geq |V_{n+1}^B|$, and set $c_{n+1} := 0$. Arbitrarily choose a value $e_{n+1} \in Z^+$.

For each $i = 1, \ldots, n+1$, choose a positive integer m_i, and set $a_i := a_i + m_i$ and $b_i := b_i + m_i$, so that $\left\lfloor \frac{a_i+b_i+c_i}{e_i} \right\rfloor$ is the same for all i (hence the states satisfy proportionality) and so that assumption A3 is satisfied. Thus we obtain a vote trading problem that satisfies proportionality and it is easy to see that any incidence vector of won states for an optimal solution to this problem is an optimal solution to our original knapsack problem. ■

Corollary 11 *The vote trading problem for S^* is NP-hard.*

Proof. The vote trading problem considered in the Proposition is a special case of a vote trading problem for S^*. ■

5 Uncertainty

The model for vote trading presented in Section 2 is deterministic. In reality, there are uncertainties in the inputs; the purpose of this section is to address this issue. Two key sources of uncertainty are estimates of how voters will vote (assumption A3) and how much cheating traders will engage in (assumption A5). In this section we consider these two sources of uncertainty and address the general problem, in S^*, of choosing signed-up voters to instruct to switch so that the *probability* of B winning the election is maximized.

We present two methods for addressing this general problem. The second method directly addresses this problem and is presented in Section 5.3. This method, however, is a bit unwieldy to implement, hence we first present, in Section 5.1, a method that is simpler, from an implementation standpoint. The simpler method requires the solution of a sequence of deterministic vote trading problems, as defined earlier in the paper. Both methods rely on a calculation of the probability of B winning a state as a function of the number of voters that are instructed to switch (from C to B or from B to C), which is the topic of Section 5.2

Our methods require two inputs in each state: standard polling data and an estimate of the number of cheaters who have signed up. Although precisely estimating the level of cheating is probably impossible, the models can help vote traders address several types of questions. For example, if vote trading has been implemented and it is time T_e, the models can help with the following type of question: What is the "best case" probability of vote trading working, assuming there is no cheating? If this probabiltiy is very low, then it may not be worth the effort to instruct voters to switch. If it has been decided to pursue such instructions, then the aggregate opinion of the vote trading organizers concerning the level of cheating can be used in the models to find a reasonable solution to the vote trading problem. Finally, before vote trading is implemented, one can address this sort of question: What is the minimum percentage of participation by C voters in swing states that would be necessary for vote trading to increase the probability of B winning by, say, two percentage points, assuming there is no cheating (and sufficient numbers of voters for B sign up in lost states)? If this percentage is high, then it may be decided that vote trading is not worth pursuing in the first place.

In Section 6 we discuss how our these methods can easily be extended to develop a strategy for cheaters.

5.1 Notation and Method 1

In this section we introduce some notation and our first method for solving the vote trading problem with uncertainty. Roughly speaking, the method searches for the largest probability q, together with a feasible set of trades to instruct, so that the set of states won with probability at least q is a winning solution for B (that is, one in which B wins a majority of the electoral votes).

Let us assume it is time T_e. We use the following notation in each state i:

- V_i^C denotes the set of signed up voters in the state who claim to be willing to switch from C to B. V_i^B denotes the set of signed up voters in the state who claim to be willing to switch from B to C. Some of these sets may be empty (if some voter types were not invited to sign up at the Web site or if voters simply did not sign up).

- l_i denotes an estimate of the number of legitimate vote traders in V_i^C (i.e., those voters who will switch their votes if so instructed). Although, for simplicity, we assume here that the l_i are fixed numbers, we could use probability distributions. The analysis given below can be altered accordingly in a straightforward manner. We are not concerned here about cheaters among the voters in the sets V_i^B, since such cheating only helps B.

- For $n_i \in \left[0, \left|V_i^C\right|\right]$, let $p_i(n_i)$ denote the probability that instructing a randomly selected subset of V_i^C of size n_i (and no members of V_i^B) to switch will win the state. For $n_i \in \left[-\left|V_i^B\right|, 0\right]$, let $p_i(n_i)$ denote the probability that instructing a randomly selected subset of V_i^B of size $-n_i$ (and no members of V_i^C) to switch will win the state. In Section 5.2 we discuss how to estimate $p_i(n_i)$ as a function of polling data and estimates l_i.

Observe that $p_i(n_i)$, could be effectively zero if A is significantly ahead in state i or if $\left|V_i^C\right|$ is small. However, $p_i(n_i)$ should be nondecreasing as n_i increases. Furthermore, for n_i fixed, $p_i(n_i)$ should be nondecreasing as l_i increases.

Let us assume, for now, that we know $p_i(n_i)$ for each state i. Let q be an arbitrary probability. For each state i, if $\min\left\{n \in \left[-\left|V_i^B\right|, \left|V_i^C\right|\right] : p_i(n) \geq q\right\}$ is well defined, then set $n_i(q)$ to be this value and call this state *q-winnable*. Otherwise call this state *q-lost*. So, suppose $n_i(q)$ is well defined. Then, when $n_i(q) > 0$, B has a probabilty less than q of winning (before switching) and $n_i(q)$ is the smallest number of voters to instruct to switch from C to B to raise this probability to at least q. And, when $n_i(q) < 0$, B has a probabilty greater than q of winning and $-n_i(q)$ is the largest number of voters to instruct to switch from B to C to keep this probability at least q.

We define the *vote trading problem for* q as follows: When i is q-winnable and $n_i(q) \geq 0$, set $a_i := 2n_i(q) + 1$, $b_i := n_i(q) + 2$, $c_i := n_i(q)$; note that when $n_i(q) = 0$, in this vote trading problem, B is in the lead in state i; otherwise, B needs $n_i(q)$ more votes to win (with probability q). We assume all voters for C have signed up to switch. When i is q-winnable and $n_i(q) < 0$, set $a_i := M + n_i(q)$, $b_i := M$, $c_i := 0$. We have chosen $M \geq \left|V_i^B\right|$ to be a suitably large number such that if $-n_i(q)$ votes are switched from B to C, then B still wins (with probability at least q). We assume $\left|V_i^B\right|$ voters for B have signed up to switch. When i is q-lost, set $a_i := \left|V_i^B\right| + 1$, $b_i := \left|V_i^B\right|$, and $c_i := 0$. We assume all voters for B have signed up to switch. Also note that the values have been chosen so that the assumptions in A3 are satisfied: i.e., $a_i > c_i$ and $b_i > c_i$. Note that the numbers a_i, b_i, c_i do not directly represent how people are expected to vote in state i (although there is an indirect link through the function $n_i(q)$).

We can now describe our first solution method for the traders. For any value of q, the traders can set up and solve the vote trading problem for q (e.g., using the associated knapsack problem described in Section 4). A winning solution for B, in the vote trading problem for q, is a solution in which B wins a majority of the electoral votes. Such a winning solution has the property that if the same numbers of vote trades are instructed in the real states, then every state that is won in the model is won in reality with probability at least q. Thus the solution method is for the traders to search for the maximum value q such that the vote trading problem for q has a winning solution for B (or discover there is no such value). In practice the maximum value q can be approximated by doing a binary search on a reasonable finite set of possible values for q.

5.2 Estimating $p_i(n_i)$

Let us now address the question of estimating $p_i(n_i)$ for each state i. To simplify our notation, let us drop the subscripts and assume we are considering an arbitrary state.

We introduce the following notation and definitions:

- For an integer $x \in \left[-\left|V^B\right|, l\right]$, let $M(x)$ be a lower bound on the probability that x legitimate vote switches (from C to B, if $x \geq 0$; from B to C, otherwise) will win the state for B. All switches from

B to C are considered legitimate, since cheating here only helps B. We discuss how to estimate $M(x)$ below.

- For integers $x \in [0, l]$ and $n \in \left[0, \left|V^C\right|\right]$, let $p(n, x)$ be the probability that a randomly selected subset of V^C of size n contains exactly x legitimate voter switchers. We discuss how to calculate $p(n, x)$ below.

There are two basic situations to consider for computing $p(n)$: when $n \geq 0$ and when $n < 0$. When $n \geq 0$ we are switching votes from C to B, and we must include in our calculations the additional uncertainty due to cheating. However, when $n < 0$, we are switching votes from B to C, and cheating is not an issue; if we assume no cheating, then we obtain a (worst case) lower bound on the probability of winning (i.e., cheating actually helps B in this case). Using the above notation, we obtain the following:

When $n \geq 0$, we have that

$$p(n) = \sum_{x=0}^{\min(l,n)} p(n, x) \cdot M(x).$$

To see this, consider the experiment of randomly selecting a subset of V^C of size n. The outcomes of this experiment are the possible numbers x of legitimate voters in the subset, which are integers in $[0, \min(l, n)]$, and whether switching them results in a win. For each such outcome we know the probability of finding this number of legitimate voters in our subset and the probability that switching this number of legitimate voters results in a win for B. Hence we obtain the above expression.

When $n < 0$, we simply set $p(n) := M(n)$. We next consider the details of how to compute $M(x)$ and $p(n, x)$. We begin with a bit more notation:

- We are given polling data from a sample of voters in the state: Let P^* denote the subset of the sample that prefers A or B; and let $\overline{p}_A$ and $\overline{p}_B$ denote the proportions of P^* that prefer A and B, respectively (these numbers are easily obtained from a standard poll that also contains results for other candidates).

- Let p_A and p_B denote the true proportions of voters in the state who will be voting for A and B, respectively, taken from the set of voters who will be voting for A or B.

- Let P denote an estimate of the number of people who will vote for either A or B (in particular, choose a minimum value P so that, with high probability, the number of people who will vote for either A or B is $\leq P$).

To begin, for any probability p^* we can make (approximate) confidence intervals for p_A and p_B using the standard formulas (see Anderson et al (2004), for example)

$$\overline{p}_A \pm E(p^*)$$
$$\overline{p}_B \pm E(p^*)$$

where $E(p^*) = z_{\frac{1-p^*}{2}}\sqrt{\frac{\overline{p}_A\overline{p}_B}{|P^*|}}$. Thus p_A and p_B are contained in the respective intervals with probability p^* (approximately). By multiplying these formulas by P, we get (approximate) confidence intervals for the number of actual votes that will go to each candidate in the state:

$$P\cdot\, \overline{p}_A \pm P \cdot E(p^*)$$
$$P\cdot\, \overline{p}_B \pm P \cdot E(p^*)$$

Suppose $\overline{p}_A \geq \overline{p}_B$. If $x \leq P \cdot \overline{p}_A - P \cdot \overline{p}_B$, then $M(x) = 0$. If $x > P \cdot \overline{p}_A - P \cdot \overline{p}_B$, then $M(x)$ is equal to the value of p that satisfies: $x = [P \cdot \overline{p}_A + P \cdot E(p)] - [P \cdot \overline{p}_B - P \cdot E(p)]$.

Suppose $\overline{p}_B > \overline{p}_A$. If $x \leq P \cdot \overline{p}_A - P \cdot \overline{p}_B$, then $M(x) = 0$. (Note that in this case x is negative, hence we are switching votes from B to C.) If $x > P \cdot \overline{p}_A - P \cdot \overline{p}_B$, then $M(x)$ is equal to the value of p that satisfies: $P \cdot E(p) = \frac{1}{2}\left(\{P \cdot \overline{p}_B + x\} - P \cdot \overline{p}_A\right)$.

Using these observations, we obtain a method to compute $M(x)$. Finally, we consider how to calculate $p(n, x)$.

In practice, we do not know which of the signed up voters in the state are legitimate; we only know (actually have an estimate) that of the set V^C of voters who have signed up, l are legitimate. However, for a randomly selected subset of V^C of size n, we can calculate the probability, that is $p\,(n, x)$, that the subset contains exactly $x \leq \min(l, n)$ legitimate voters by using the hypergeometric distribution (see Anderson et al (2004)) as follows:

$$p\,(n, x) = \frac{\binom{l}{x}\binom{|V^C|-l}{n-x}}{\binom{|V^C|}{n}}.$$

5.3 Method 2

Finally, let us directly address the question of finding a solution to the vote trading problem that maximizes the probability of B winning. To begin, consider the problem of calculating the probability of B winning for an arbitrary feasible solution to the vote trading problem. Using this feasible solution, and the techniques described above for calculating $p_i(n_i)$, we can calculate for each state the probabilities of B winning and losing. If we assume these probabilities are independent across the states, then we can calculate the probability of any particular combination of wins and losses of states (by taking the product of the probabilities over the states). And by adding the probabilities for those combinations that win the election for B, we obtain the probability of B winning. Thus, by searching over all possible feasible solutions to the vote trading problem, we can find a solution that yields the maximum probability of B winning. Good solutions can be sought using, for example, a gradient search heuristic over the space of feasible solutions to the vote trading problem.

6 A Strategy for Cheating

In this brief section we informally discuss a strategy for cheating. The omitted details are a straightforward variation on the analysis given for the trader's problem in the previous section.

Let us begin by discussing some assumptions. Let us assume the traders are using the optimal system S^* and let us assume the cheating is directed by a centralized organization (any "rogue" cheating only helps the cheaters and increases the probability of A winning). Let us assume the cheaters have a limited number of times they can sign up at the vote trading Web site (due to a limited number of participants and the security measures used by the traders). Let us also assume that the cheaters adopt the straightforward strategy of signing up at the traders' Web site as C voters in swing states, and then, if asked to switch, voting for A in the election. Finally, let us assume the cheaters have access to the same polling data that the traders have.

Thus, the general problem for the cheaters is to distribute their limited number of cheating sign ups across the states in such a way that the probability of winning for B is minimized (or the probability of winning for A is maximized). Let t_i be a variable denoting the number of cheating sign ups in each state i (where the sum of the t_i is the total number of cheating sign ups available). The difficulty for the cheaters is that they do not know how many legitimate vote traders have signed up in each state. So they must estimate this. Let us say m_i is their estimate of the number of legitimate vote traders signed up at each state i. Now, for any solution t, the cheaters can reproduce the calculation the traders would use (from Section 5.1 or 5.3) if they correctly guessed the vector t (which is designated l in their model). Thus the cheaters can compute the worst case (from their point of view) probability q of B winning for any vector t (given their estimates m). Hence their objective is to find the vector t that minimizes the probabilty q of B winning (in the sense used in Section 5.1 or 5.3). Reasonable solutions can be sought using search heuristics.

7 Assumptions, Extensions, and Open Problems

In this section we discuss several of the assumptions from Section 2, we discuss some extensions of ideas of the paper, and we mention some open problems.

In assumptions A1 and A4 we have limited consideration to vote trading between two candidates in a race with three significant candidates. This nicely mirrors the situation in the 2000 and 2004 Presidential elections where there were two main candidates, A and B, and a significant minor candidate C whose positions were closer to those of B than A. If C's positions were more midway between the main candidates, then supporters of both A and B might have an interest in using vote trading strategies. The same situation could hold if there were two main candidates, A and B, and two significant minor candidates, C and D, whose positions were closer to A and B, respectively. These variations appear to be more complex to analyze than the situation addressed in this paper and would seem to require the application of the techniques of game theory.

In another variation there are two main candidates, A and B, and two (or more) significant minor candidates, say C and D, whose positions are closer to those of B. The results of Sections 3 and 4 generalize easily (in fact, this general situation was exploited in 2004; see VotePair.org).

In another variation there are three main candidates, A, B, and C, where A and C have similar positions. The objective of vote trading is to elect either candidate A or C. Consider two types of states: states where A and B are neck and neck and C is out of contention and states where B and C are neck and neck and A is out of contention. Vote trading can occur between C supporters in the first type of state and A supporters in the second type of state. Vote trades are sought that increase the number of electoral votes for both A and C as much as possible. The type of analysis in Sections 3 and 4 can be done for this situation.

In a slight variation on the last example there are three main *parties*, A, B, and C, where A and C have similar positions (this was the situation in the U.K. Parliamentary elections in 2001; see Ledbetter (2001)). The objective is to have as many candidates from A and C win district elections as possible. We consider two types of districts: districts where the candidates from parties A and B are neck and neck and the candidate from C is out of contention and districts where the candidates from parties B and C are neck and neck and the candidate from A is out of contention. Vote trading can occur between C supporters in the first type of district and A supporters in the second type of district. The payoff in this variation for winning a district is always one seat, hence the problem is simpler than the electoral college system, where varying numbers of electoral college votes are at stake. The results from Section 3 carry over easily, but the corresponding version of the vote trading problem can be seen to be polynomially solvable.

Assumptions A4 and A7 can be generalized by allowing Web site invitations to be made to arbitrary types of voters, and allowing this to vary by state. Also, Web sites may be allowed to operate in different time windows. Allowing such generalizations into the model has no effect on the main result in Section 3 concerning the "optimal" vote trading system (the analysis must be elaborated, but it follows the same lines given).

Finally, we observe that the model in Section 5 can be elaborated by setting different target probabilities for each state. For example, one may want to have a higher relative probability of winning states with a large number of electoral votes. The procedure given can easily be adapted to this variation.

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