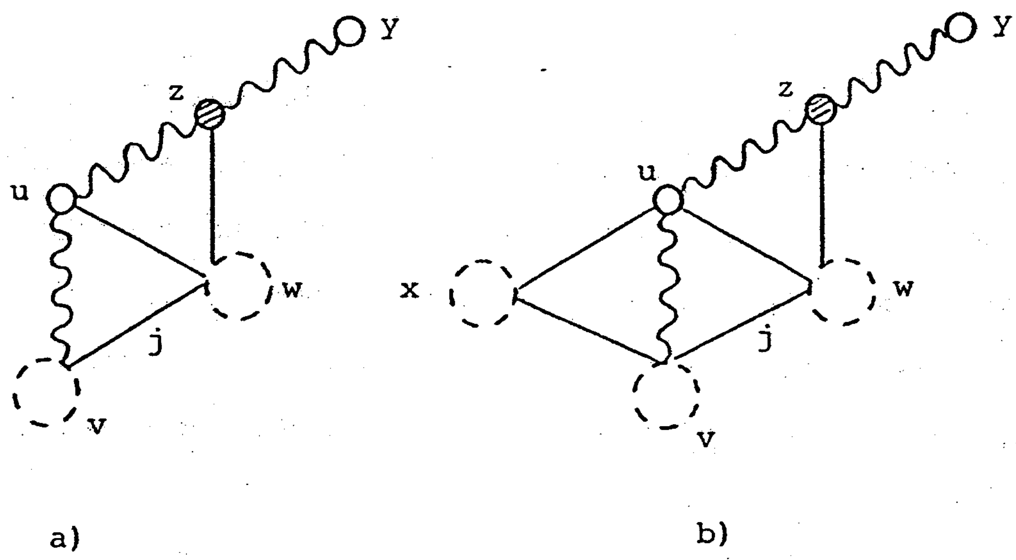

Figure 3.15

Note that u must be real even since z is dummy odd. Therefore v must be shrunk even and p(u) = v. Since p(z) = w, v is the only node which has a predecessor which is not among the other nodes (it may be a root).

In case a) make z an odd node of $\mathcal{S}$. If y is not in $\mathcal{S}$ or is a dummy odd node of $\mathcal{S}$, make y an even node of $\mathcal{S}$ with $P_y = (y,z,w,P_v)$. Make uvw a triangular petal and no longer call uwz a triangular petal. Go to Step 2.

In case b) set $P_z = (z,u,w,v,u,x,P_v)$, $P_y = (y,z,w,P_v)$, S = (u,v,w,x,z), b = v, and go to Step 5.

Case 3.10c: uvw shares edges with one or two triangular petals uvx and/or uwz where all four or five nodes are even. We have the following possibilities:

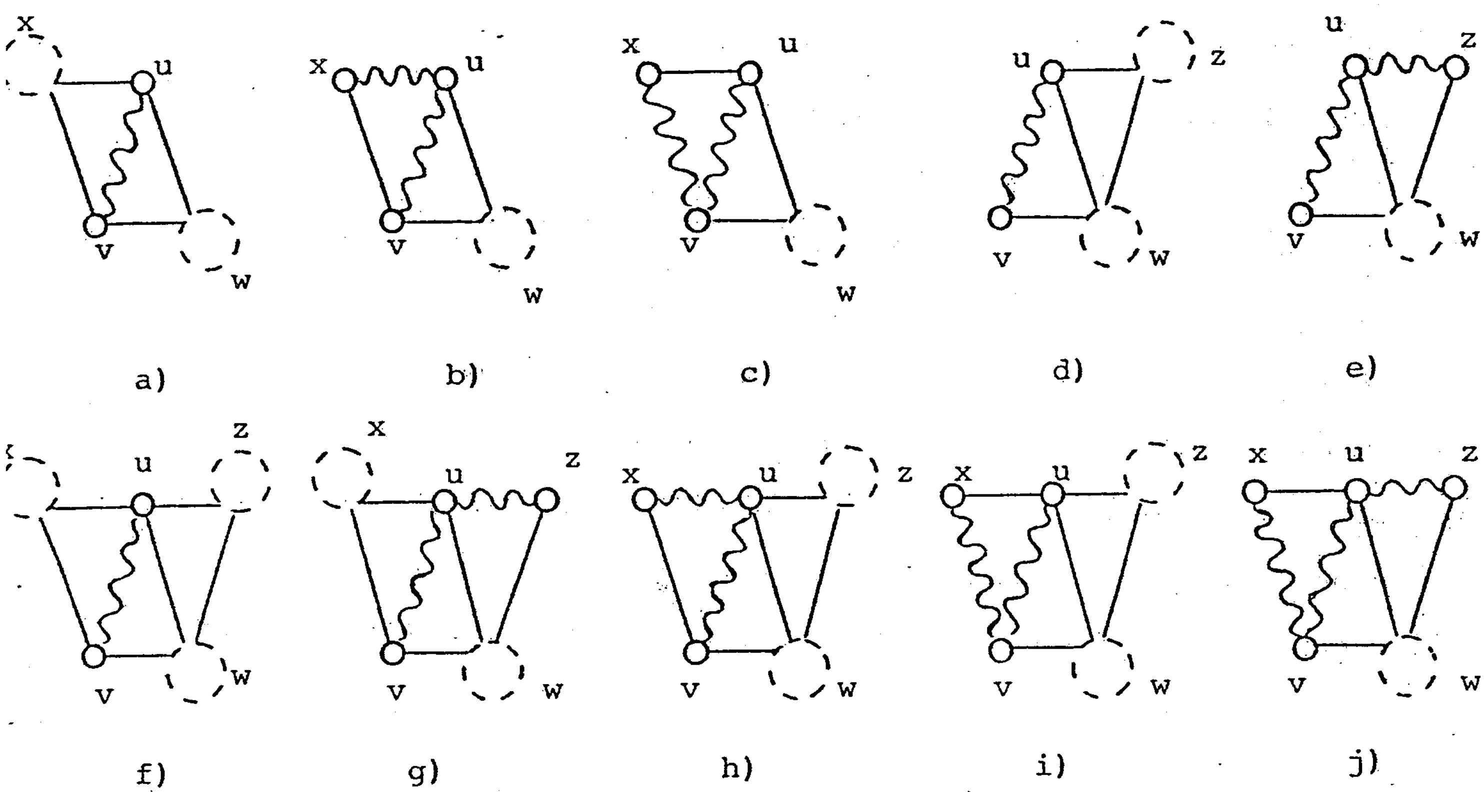

Figure 3.16

There are one or two nodes in each case whose predecessors are not among the other nodes and they must be nodes which have degree 1 in the matching in the figure (same reasoning as in Case 3.8c).

Assume for each case there are two nodes whose predecessors are not among the others (they may be roots). Using the following, we go to Step 3.6 and either shrink or augment.

a) and f): Set $P_{vu} = (v,x,u,v,w,u)$

b) and h): Set $P_{xv} = (x,v,u,w,v)$

c) and i): Set $P_{xu} = (x,u,v,w,u)$

d) Set $P_{vu} = (v,w,z,u)$

e) and g): Set $P_{vz} = (v,w,z)$

j) Set $P_{xz} = (x,u,v,w,z)$.

Assume for each case there is just one node whose predecessor is not among the others. Then we may reason as in Case 3.8c to see that a shrinking was being blocked because a matching edge with one endnode y, which is adjacent to one of the four or five nodes in the figure, could not become a petal. We consider all the possible cases:

a) petal yv, $P_y = (y,v,x,u,v,w,P_u)$, $S = (u,v,w,x)$, $b = u$

petal yu, $P_y = (y,u,w,v,u,x,P_v)$, $S = (u,v,w,x)$, $b = v$

b) petal yv, $P_y = (y,v,w,u,v,P_x)$, $S = (u,v,w,x)$, $b = x$

c) petal yu, $P_y = (y,u,w,v,u,P_x)$, $S = (u,v,w,x)$, $b = x$

d) petal yu, $P_y = (y,u,z,w,P_v)$, $S = (u,v,w,z)$, $b = v$

f) petal yu, same as for a)

i) petal yu, same as for c).

Go to Step 5.

<u>Step 3.11</u> [w is even and j forms a blocking triangle with no matching edge which does not share an edge with an existing petal].

Let uvw be the blocking triangle. It must be the case that two of the three nodes of the blocking triangle are even shrunk nodes. The third node is real even (otherwise the triangle is not blocking and we are in Case 3.6).

Call the triangle uvw a triangular petal and go to Step 2.

Step 3.12 [w is even and j forms a blocking triangle which has no matching edges and which shares an edge with one or two existing triangular petals].

Let uvw be the blocking triangle containing the edge j. We have the following possibilities for uvw:

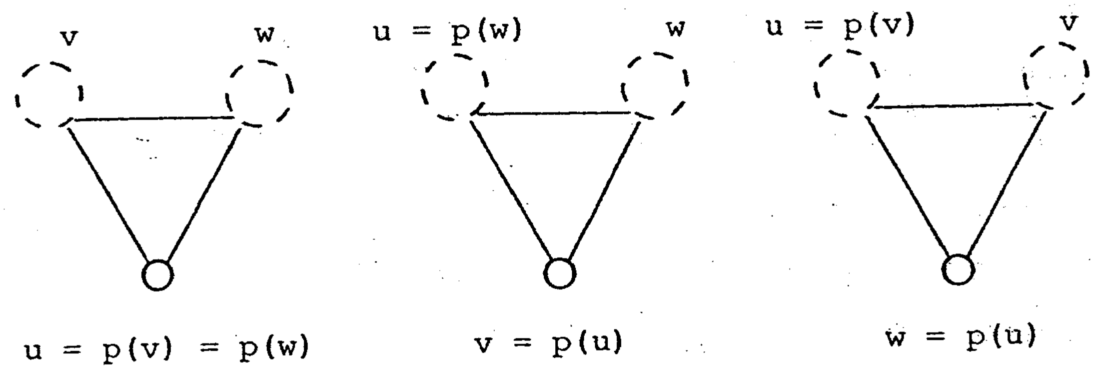

Figure 3.17

Since v and w play symmetrical roles, let us assume, without loss of generality, that w is shrunk.

Case 3.12a: There exists a triangular petal uvx where x is dummy odd. We have the following possibilities:

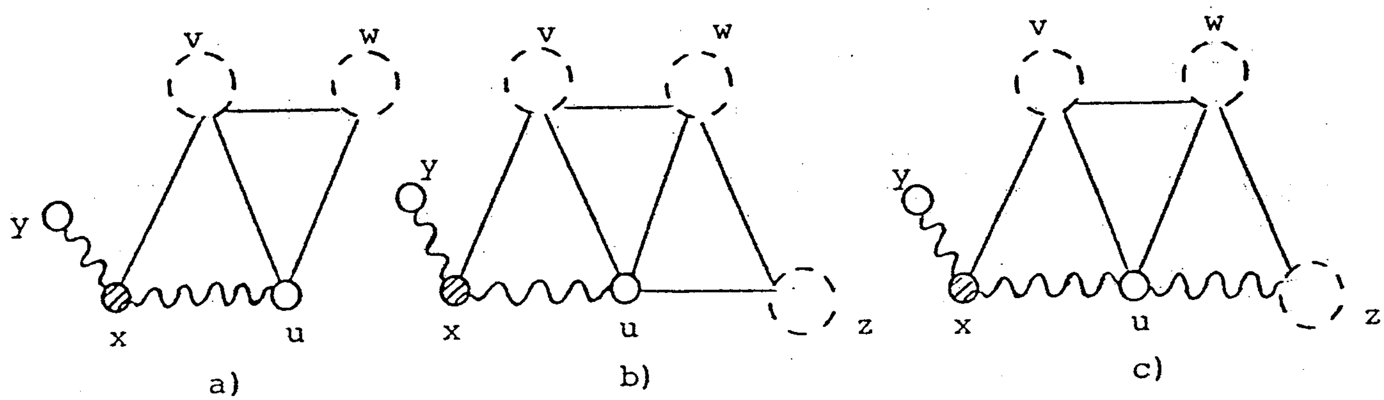

Figure 3.18

Note that u and v may be interchanged in these examples. The corresponding cases can be handled by interchanging u and v in what follows.

In all cases u must be real even since x is dummy odd. Also, in all cases p(x) = v. In a) and b) u is the only node whose predecessor is not among the other nodes. In c) p(u) = z, so in this case z is the only node whose predecessor is not among the others.

In case a) make x an odd node of $\mathcal{S}$. If y is not in $\mathcal{S}$ or is a dummy odd node of $\mathcal{S}$, make y an even node with $P_y = (y,x,v,w,P_u)$. Make uvw a triangular petal and no longer call uvx a triangular petal. Go to Step 2.

In case b), set $P_x = (x,u,v,w,z,P_u)$, $P_y = (y,x,v,w,P_u)$, S = (u,v,w,x,z), b = u, and go to Step .

In case c), set $P_x = (x,u,v,w,P_z)$, $P_y = (y,x,v,w,P_z)$, S = (u,v,w,x,z), b = z, and go to Step 5.

Note that uvx cannot have two matching edges since one of them would be incident with a shrunk node u or v. This implies that the base of this shrunk node is a matching edge, which is not the case. Thus we have just the remaining one case.

<u>Case 3.12b</u>: uvw shares edges with one or two triangular petals uvx and/or uwz and all four or five nodes are even. We have the following possibilities:

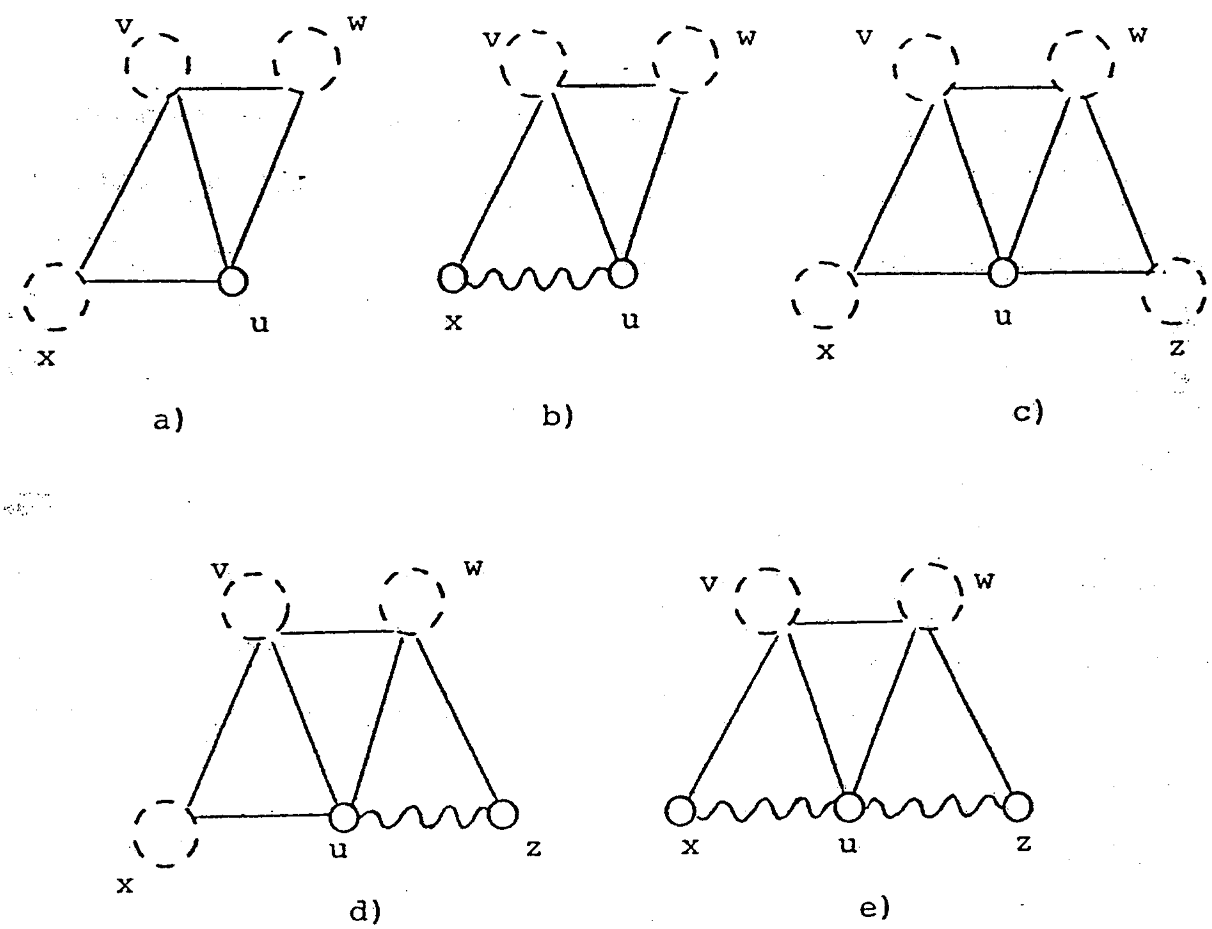

Figure 3.19

Note that we can interchange the roles of u and v in a) and b) and interchange the roles of v and w in d) to get three more cases. These cases can be dealt with in what follows by making the corresponding interchanges.

In cases a) and c), u is the only node whose predecessor is not among the other nodes. If u is deficient by 2, then an augmentation was being blocked. Set $P = (u,x,v,w,u)$ and go to Step 4. If u is deficient by 1 or saturated, then a shrinking was being blocked. If u is saturated, then there is an edge $yu \in M$ which could not become a petal. Set $P_y = (y,u,x,v,w,P_u)$. In either case, set $S = (u,v,w,x)$ or (u,v,w,x,z) and go to Step 5.

For the remaining cases, assume there are two nodes whose predecessors are not among the other nodes (they may be roots). Using the following, we go to Step 3.6 and either shrink or augment.

b) Set $P_{xv} = (x,u,w,v)$

d) Set $P_{uz} = (u,v,w,z)$

e) Set $P_{xz} = (x,v,w,z)$.

Finally, we assume for these cases that there is just one node whose predecessor is not among the others. Then, reasoning as in Cases 3.8c and 3.10c, we had a blocked petal and shrinking. There are only two possible cases:

b) petal yu, $P_y = (y,u,w,v,P_x)$, $S = (u,v,w,x)$, $b = x$

d) petal yu, $P_y = (y,u,v,w,z)$, $S = (u,v,w,x,z)$, $b = z$.

Go to Step 5.

Step 3.13 [w is a dummy odd node]: Suppose u_1w and $u_2w \in M$ and u_1wx is the triangular petal containing w. Hence the addition of the edge u_2w to $\mathcal{S}$ and the making of u_2 into an even node (assuming it was dummy odd or not in $\mathcal{S}$ when the petal containing w was formed) was blocked by the triangular petal u_1wx. The objective in this step is to make u_2 even, if it is not already even, and to make w an odd node of $\mathcal{S}$. We will see that this can be done in all cases. In particular, if u_2 is even, make w an odd node of $\mathcal{S}$ and go to Step 2. So let us assume u_2 is dummy odd or not in $\mathcal{S}$. Note that in all cases u_1 is real even. We have a blocking triangle if augmenting on $P_{u_2} = (u_2,w,P_v)$ creates a triangle. So if we do not have a blocking triangle, make w an odd node, make u_2 an even node, and set $P_{u_2} = (u_2, w, P_v)$. No longer call u_1wx a triangular petal. If x is a real node, remove the matching edge xu_1 from $\mathcal{S}$. Suppose x is shrunk. If $u_1x \in M$, make u_1x a petal of x. If $u_1x \notin M$, make u_1x the base of x. Go to Step 2.

Let us assume we do have a blocking triangle. If it has two matching edges, then the two edges must be u_1w and u_1v; if it has one matching edge, then the edge must be u_1w. Thus if u_1wx has two matching edges and we form a blocking triangle with two matching edges, then $v = x$. So we disregard this case. Note also that it is not possible here to form a blocking triangle with no matching edges. The remaining cases are illustrated in Figure 3.20.

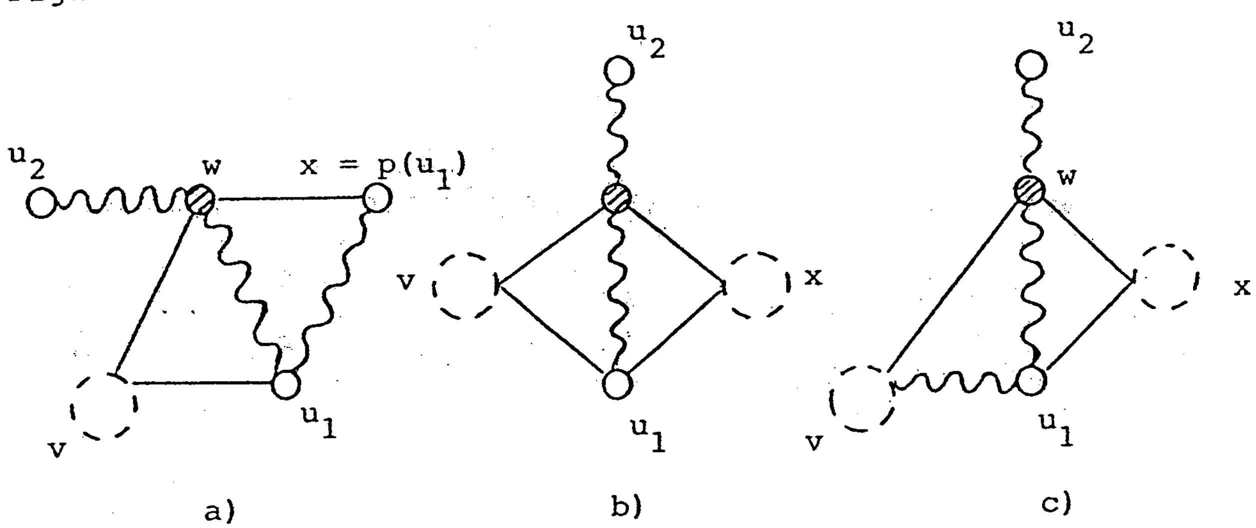

Figure 3.20

Note that the blocking triangle u_1vw may share a second edge, that is u_1v, with an existing petal, call it u_1vz. We illustrate these possibilities in Figure 3.21.

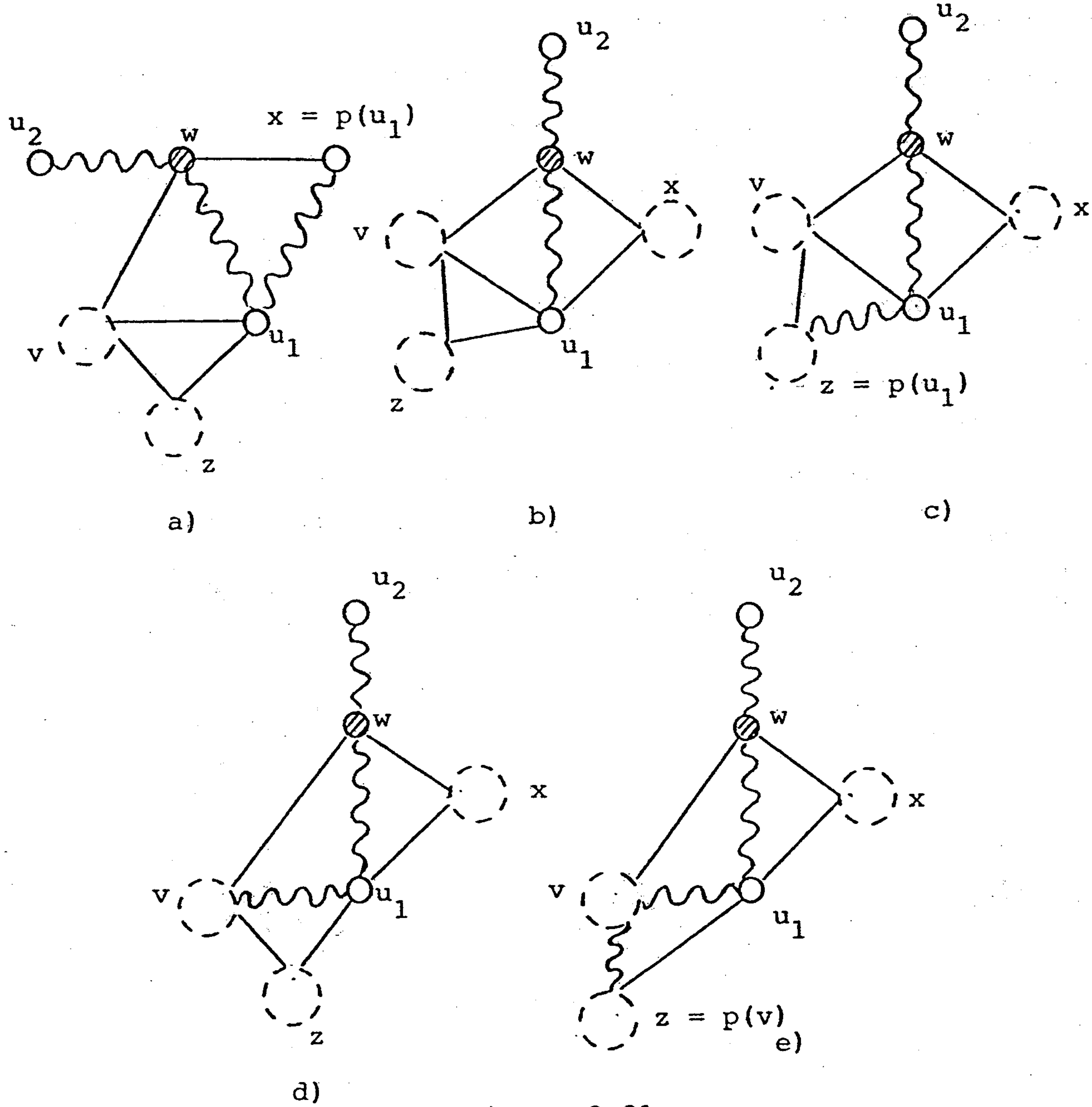

Figure 3.21

Note that in all cases in Figures 3.20 and 3.21, v and z are shrunk even. We set P_{u_2} as follows:

Fig. 3.20 a) and Fig. 3.21 a): $P_{u_2} = (u_2, w, v, u_1, w, P_x)$

Fig. 3.20 b) and Fig. 3.21 b) and c): $P_{u_2} = (u_2, w, x, u_1, w, v, P_{u_1})$

Fig. 3.20 c) and Fig. 3.21 d) and e): $P_{u_2} = (u_2, w, x, u_1, w, P_v)$.

Make w an odd node and make u_2 an even node. No longer call u_1wx a triangular petal. (However, u_1vz remains a triangular petal.) If x is a real node, remove the matching edge xu_1 from $\mathcal{S}$. Suppose x is shrunk. If $u_1x \in M$, make u_1x a petal of x. If $u_1x \notin M$, make u_1x the base of x. Go to Step 2.

Step 4 [Augmentation]: Let e be a generic edge of P. If $e \notin M$, then add the edge e to the set M; on the other hand, if $e \in M$, then remove the edge e from M. Let u be a generic shrunk node encountered on the path P. Modify the matching inside u so that the new matching is a triangle-free simple 2-matching. (This modification of M inside u involves interchanging edges in and out of M along a portion of the alternating cycle defined when u was shrunk. The fact that the modification is possible follows from the validity of the shrinking step. See Theorem 3.1.) Throw $\mathcal{S}$ away and go to Step 1.

Step 5 [Shrinking]: We are given S and b.

Shrink all the nodes of S into a single node of $\tilde{G}$, say node u.

Case 5a: If S contains a node which is deficient in G, then call u a root of $\mathcal{S}$. Let $T = M \cap \delta(S)$. Continue Step 5.

Case 5b: If S does not contain any deficient node of G, then let $j = bp(b)$. Call j the base of u. If $j \in M$, let

$T = (M \cap \delta(S))\backslash\{j\}$ and if $j \notin M$, let $T = (M \cap \delta(S)) \cup \{j\}$. Define P_u as P_b. Continue Step 5.

Let the petals of u be

(i) the triangular petals of $\mathcal{S}$ which contain one node of S, and

(ii) the edges of T which do not belong to triangular petals of $\mathcal{S}$ (these edges form the edge petals of u).

If, in some triangular petal of u, the two other nodes, say x and y, are shrunk nodes of $\mathcal{S}$, then define S as the set of nodes comprised in u, x and y. At least one of the paths P_u, P_x, P_y does not contain the two other nodes, say P_b. Go to Step 5 with this new S and b.

For any triangular petal uxy such that x is a shrunk even node and y is a dummy odd node, change y into an even node of $\mathcal{S}$ and set $P_y = (y, P_u)$. (This is valid since $yu \in M$ and $yx \notin M$, see Figure 3.4.)

Check whether some edge petal t of u joins u to an odd node of $\mathcal{S}$ or to a shrunk even node of $\mathcal{S}$. If there exists such an edge $t = (u', w)$, $u' \in S$, where $p(w) \neq u'$ and $p(u) \neq w$, then go to Step 3.6 where v and w are taken to be the two endnodes of t.

Check whether some nonmatching edge of $\mathcal{S}$ joins u to a node w of $\mathcal{S}$ where $p(w) \neq u'$, $u' \in S$, and $p(u) \neq w$. If such an edge exists, set v and w to be the endnodes of that edge. Go to Step 3.6.

For each edge petal tip z which is not in $\mathcal{S}$, make z an even node of $\mathcal{S}$, define $P_z = (z, P_u)$ and add the edge uz to $\mathcal{S}$.

For each edge petal tip z which is a dummy odd node of $\mathcal{S}$, make z an even node of $\mathcal{S}$ and let $P_z = (z, P_u)$. Add the edge uz to $\mathcal{S}$.

For each odd node w of $\mathcal{S}$ such that $p(w) \in S$, reset $p(w) = u$. Similarly for each shrunk even node w of $\mathcal{S}$ such that $p(w) \in S$, reset $p(w) = u$.

Finally, if two petals k and ℓ of u have a common even node z as a tip, and neither k nor ℓ contains a dummy odd node, then modify the blossom tree by adding the node z (and other tips of k and ℓ, if any) to the set S, and by removing k and ℓ from the set of petals. If either k or ℓ happens to be the edge j, then reset $P_u = P_z$. Go to Step 2.

End of Algorithm

Section 4. The Validity of the Algorithm

In order to prove that the algorithm polynomially solves the cardinality triangle-free simple 2-matching problem, we must prove the following:

Proposition 3.3: In Step 3.2 both u_1v' and $u_2v' \in M$ cannot occur.

Theorem 3.1: The shrunk nodes defined in Step 5 are critical and they are always saturated by the current triangle-free simple

2-matching M. (This implies that augmentations through shrunk nodes are valid.)

Proposition 3.4: The algorithm terminates after a polynomial number of iterations.

Theorem 3.2: When the algorithm terminates, the triangle-free simple 2-matching has maximum cardinality.

Proposition 3.3: In Step 3.2 both u_1v' and $u_2v' \in M$ cannot occur.

Proof: Let us assume u_1 is not a dummy odd node and $u_1v' \in M$. (If u_1 was dummy odd we would be in Step 3.3.) Under these assumptions we show that $u_2v' \notin M$. (See Figure 3.22.)

Assume $u_2v' \in M$. Since u_1 is not a dummy odd node, it must be a real even node or not in $\mathcal{S}$ (if it were either odd or shrunk then $w \in \mathcal{S}$). Since v is a real or shrunk even node and since u_1 is real even or not in $\mathcal{S}$, u_2 must be a dummy odd node (if it were odd then $w \in \mathcal{S}$). Dummy odd nodes are made in either Step 3.2 or 3.4.

Suppose u_2 was made a dummy odd node in Step 3.2 and that we are now facing the situation where u_1v' and $u_2v' \in M$ occurs for the first time in the algorithm. Then $vp(u_2) \in M$ which implies $p(u_2) = u_1$ since $vu_1 \in M$. This implies that $u_1 \in \mathcal{S}$ and is hence real even. When u_1u_2 was considered and u_2 was made a dummy odd node, we were in Step 3.2 with w and v' playing the roles of u_1 and u_2. Thus we had wu_1 and $v'u_1 \in M$ which contradicts our assumption that in considering vw we are facing this situation for the first time.

Suppose u_2 was made a dummy odd node in Step 3.4. Then there must be a shrunk node x such that xu_2v is a triangular petal with the one matching edge u_2v. It must also be true that u_2 was made a dummy odd node and hence entered $\mathcal{S}$ after v entered $\mathcal{S}$. But since u_1 is real even or not in $\mathcal{S}$ and v is even and saturated, it is not possible that v could have entered $\mathcal{S}$ before u_2. Contradiction.

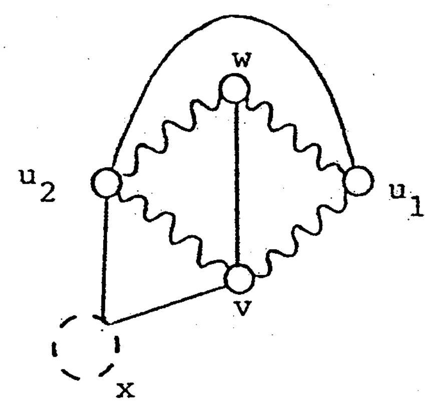

Figure 3.22

Theorem 3.1: The shrunk nodes defined in Step 5 are critical and they are saturated by the current triangle-free simple 2-matching M.

Proof: We begin the proof by observing that M saturates the blossom tree B associated with the shrunk node u. The remainder of the proof consists of proving that B is critical. We do this in two main parts. In Part 1), we exhibit augmenting paths from every node in B to the root and in Part 2), we prove that using these augmenting paths can create no triangles, which shows that B is critical.

By inductive hypothesis we assume the theorem is true for all blossom trees of $\mathcal{S}$. Note that $\mathcal{S}$ is the structure just before we go into Step 5 where we shrink S into a shrunk node u.

First, we see that M saturates B. If S does not contain the root r, then every node of S is saturated by M. In addition, by definition of the petals, M saturates B. The only deficient node of B is one of the endnodes at the base defined in Step 5. In the case that S contains r, M also saturates B, since the only deficient node in B is r (or a node of G inside r if r is a shrunk node).

<u>Part 1</u>: We now exhibit augmenting paths from every node in S and from every petal tip of u to the root. (These paths are defined much as we did in Theorem 2.1 for the simple 2-matching problem.)

If we go to Step 5 from any step other than Step 3.6, (i.e. Steps 3.8, 3.9, 3.10, 3.12 or Step 5 when we shrink three even nodes together) then all of the paths have been defined and we are done. So, let us assume that we are going to Step 5 from Step 3.6. In this case we have the two nodes v and w from Step 3.6 and a path between them which may consist of one or more edges. Let us denote this path by P_{vw}.

For every node x which is in S or is a petal of u and even, let $P = P_x$. (This includes all nodes on P_{vw}.) Suppose

x is an odd node and suppose, without loss of generality, that x occurs on P_v. Let $P = ((P_v^x)^{-1}, P_{vw}, P_w)$.

Let x be an edge petal tip of u which is not in S or is dummy odd. If x is adjacent to an odd node y of S, say in P_v, then let $P = (x, y, P_{p(y)})$. If x is adjacent to an even node y of S, where y is not contained in a blocking triangle, say $y \in P_v$, let $P = (x, (P_v^y)^{-1}, P_{vw}, P_w)$. Suppose x is adjacent to an even node y of S, where y is contained in a blocking triangle qyz where, say, P_v contains two or three nodes of qyz. See Figure 3.23.

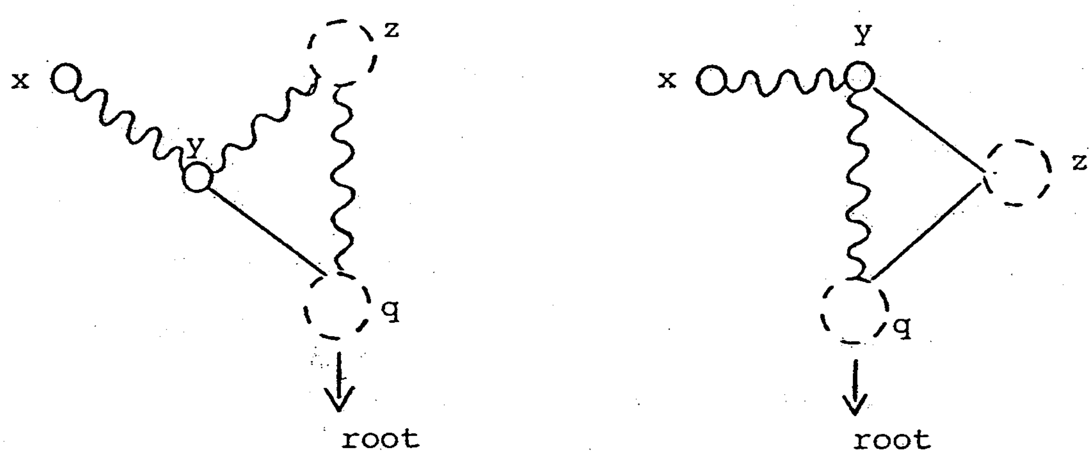

Figure 3.23

If P_v hits y first, among q, y, and z, then let $P = (x, (P_v^y)^{-1}, P_{vw}, P_w)$. If P_v hits z first and qyz has two matching edges, let $P = (x, y, q, (P_v^z)^{-1}, P_{vw}, P_w)$. If P_v hits z first and qyz has one matching edge, let $P = (x, y, (P_v^z)^{-1}, P_{vw}, P_w)$.

Suppose x is an edge petal tip of u which is adjacent to a dummy odd node of S in a triangular petal qyz where, say, P_v contains two or three nodes of qyz. Assume P_v hits z first. Then, if qyz has two matching edges, let $P = (x,y,q,(P_v^z)^{-1},P_{vw},P_w)$ and if qyz has one matching edge, let $P = (x,y,(P_v^z)^{-1},P_{vw},P_w)$.

If y is the dummy odd node in the last case, let $P = (y,(P_v^z)^{-1},P_{vw},P_w)$ when qyz has two matching edges and let $P = (y,q,(P_v^z)^{-1},P_{vw},P_w)$ when qyz has one matching edge.

Part 2: We now face the question of whether augmenting along any of these paths creates a triangle.

By inductive hypothesis, augmenting along a path P_v does not create a triangle. We must consider augmenting paths of the form (P_1,P_{vw},P_w). In particular, "Does augmenting along a portion P_1 of a path P_v ever create a triangle?" The only place in an augmenting path P_v that a triangle is ever formed is while augmenting through a structure created in Steps 3.8, 3.10, or 3.12. But, since all the nodes inside these structures (where triangles are created) are even and are incident with no new petals, we never have a path P_1 which ends inside such a structure.

So we must consider if in augmenting through (v,P_{vw},w) we create a triangle and finally we must consider if a combination of augmenting through P_1, P_{vw}, and P_w can create a triangle.

<u>Claim 1)</u>: Augmenting through (v,P_{vw},w) does not create a triangle.

If P_{vw} consists of more than just the edge vw, then P_{vw} consists of a structure as defined in Steps 3.8, 3.10, or 3.12. In none of these cases, since we have handled all cases in the algorithm, can a triangle be created. So we consider the case that P_{vw} consists of just the edge vw. If $vw \in M$, we get no triangles since the augmentation removes vw from M. If $vw \notin M$ and vw is the first edge to be considered from v to w, then, again by design of the algorithm, no triangles are formed. If, however, vw is not the first edge to be considered from v to w, then augmenting through $j = vw$ may create a triangle with one edge that is shrunk. (If two edges are shrunk, then v and w are also part of a shrunk node. We do not consider such edges j in Step 2. Figure 3.24 illustrates the possible cases. (See Figure 3.25 for some actual examples.)

<u>Claim 1a)</u>: Only four of these cases can occur.

Note that in all cases w must be even since we are shrinking. In cases c) and d), if v is a root and x is the deficient node, then zx must be a petal of v. So v cannot be a root.

(3.11) By observing all the augmenting paths defined in the algorithm and in the first part of this proof one can see that any augmenting path which contains more than one node of G inside a shrunk node of $\tilde{G}$ must contain all of these nodes consecutively along the path (i.e. the path must make a single pass through the shrunk node) and must also contain the base of the shrunk node.

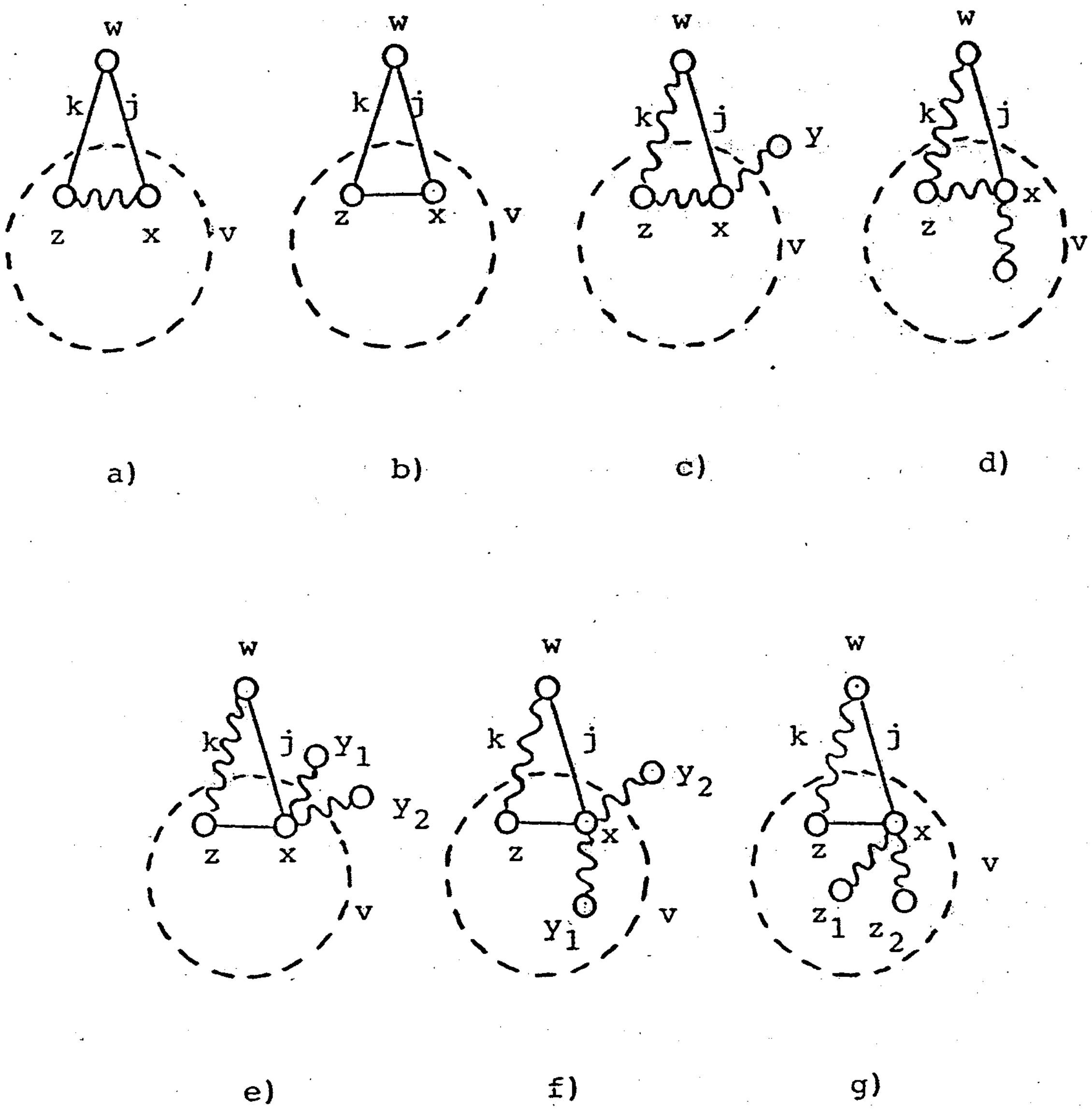

Figure 3.24

Examples:

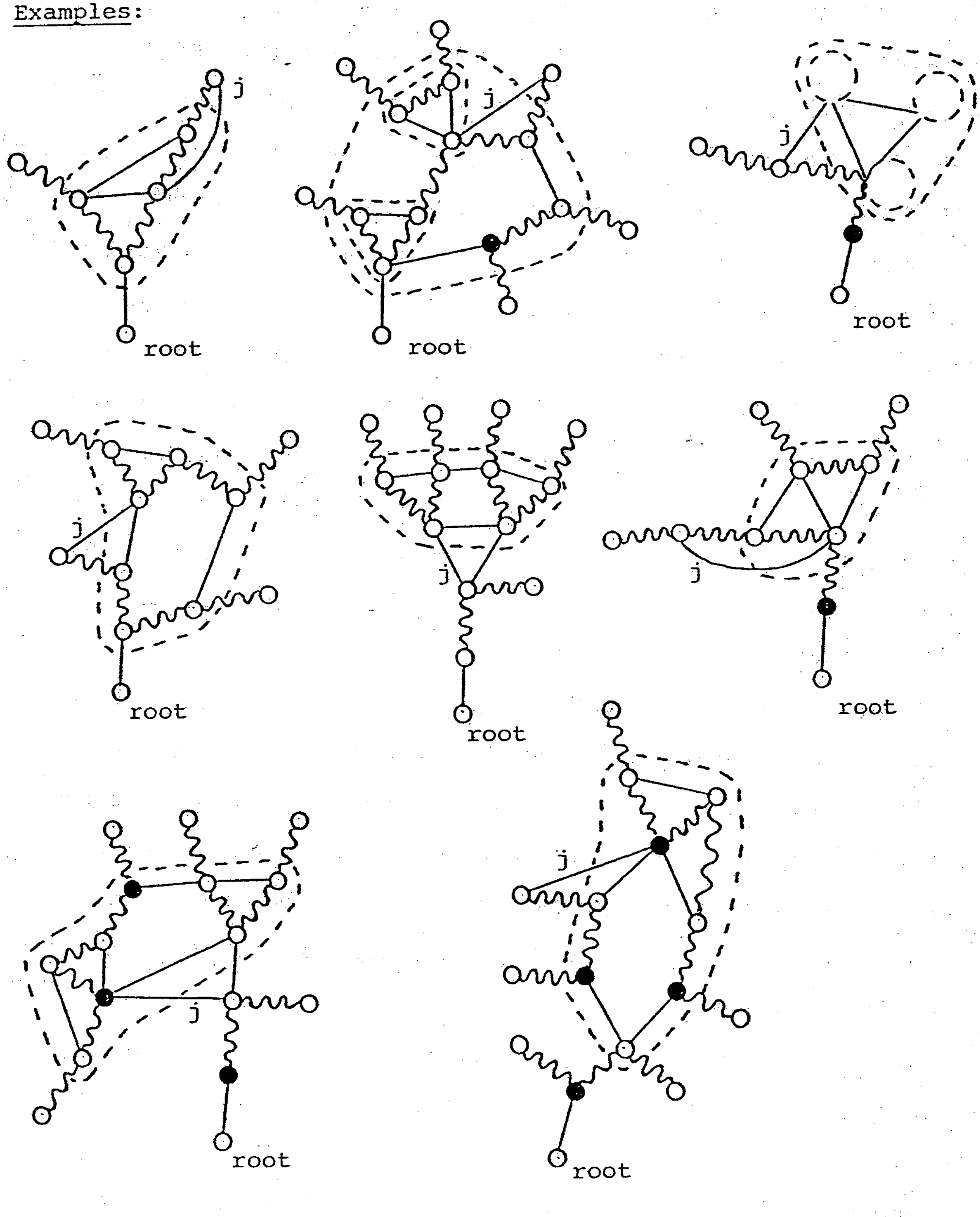

Figure 3.25

So in cases d) and g), an augmentation after edge j must contain at least two nodes of v, since both matching edges incident with x are in v. Hence such a path must exit v at its base. Therefore, in order for xzw to be blocking, k cannot be the base of v and so must be contained in a petal of v.

In cases a) and b), since w is even, the edge k must be (i) the base of either v or w or else (ii) $p(v) \neq w$ and $p(w) \neq v$ and an augmentation through k is blocked. However since we must augment through k to produce a triangle, we cannot have case (ii). In cases a) and b), if k is the base of w then it is not the base of v, and when we augment along j we cannot leave v along k, so the triangle is not blocking. So we may shrink these two nodes together, and need never use j in an augmentation.

Thus k must be the base of v. In cases a) and b) if w is shrunk, again we may just shrink these two nodes together and need never use j in an augmentation since both nodes are shrunk and petal tips are even. So the final possibility is that w is real even and k is the base of v. The only problem in shrinking v and w together in a) is that a petal wy may appear which is adjacent to w. In this case setting $P_y = (y,w,x,z,P_w)$ allows the shrinking to be performed. Thus we are left with w as a real even node and k as the base of v in case b).

Case e), by our observation (3.11), cannot be a blocking triangle since an augmenting path which creates a triangle must contain the sequence w,x,y_1 or w,x,y_2 and so must revisit the shrunk node v in order to put the edge zx into the matching.

Suppose a triangle is produced in case f). Then an augmenting path must contain the sequence of vertices $w,x,y_1,\ldots,z,x,y_2$. This implies that xy_2 is the base of v. We may shortcut such augmenting path sequences to the sequence w,x,y_2, which creates no triangle. So the remaining cases are b), where w is a real even node and k in the base of v, c), d), and g).

Claim 1b): By more carefully choosing our augmenting paths none of these cases need produce a triangle.

Suppose zx is shrunk for the first time in some node v' and suppose an edge ℓ was considered which caused the node v' to be shrunk. Let us unshrink this node by removing ℓ from $\mathcal{S}$ and renaming all the nodes in v' as they were before the shrinking was performed. If $\ell = zx$, then in cases b) and g), where $zx \notin M$, there exists an augmenting path from x to the root which does not use zx. We may use this in conjunction with j to form triangle-free augmenting paths in these two cases. In cases c) and d), where $zx \in M$, there exists an augmenting path from x through z to the root. (The cases where such an edge may create a shrinking arise in Step 5 where either z is odd and x is shrunk even, or x is odd and z is shrunk even, or x and z are shrunk even.) So let us assume $\ell \neq zx$.

Suppose x is even.

If we now consider j in Step 2, then we identify the blocking triangle xzw since all of its edges are in $\tilde{G}$. Let us now consider the edge ℓ in Step 2. We will again perform the shrinking almost exactly as before except that now we have two shrunk nodes of the blocking triangle xzw. The shrinking procedure now defines for us augmenting paths from w through j. By inductive hypothesis these paths, since they are the same as paths defined through v' earlier in the algorithm, create no triangles when passing through ℓ and so are valid.

Suppose x is dummy odd.

Then we must have one of the following two situations:

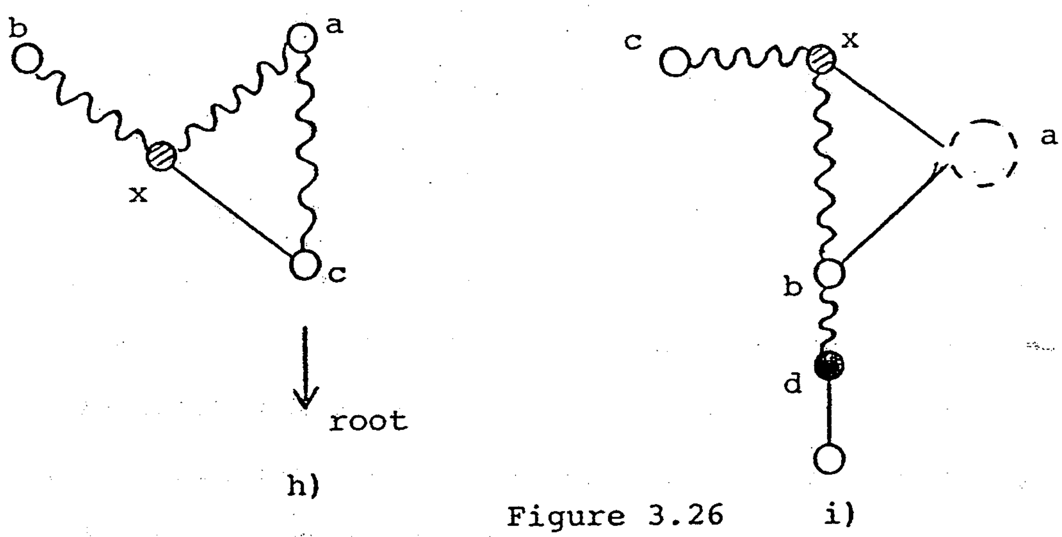

Figure 3.26

Note that in case b), if x is dummy odd then considering j in Step 2 puts us into Step 3.13 where no blocking triangles are created. So we must consider cases c), d), and g). In case h), in order for this dummy odd node to become even in a shrinking due to edge ℓ, we must have an augmenting path which contains the sequence (a,x,c). In i) we must have the sequence (a,b,d).

Let us consider case c). (Note that xy must be the base of v since it is the first edge in P_v and hence it must be contained in a petal of v.) We look in h) and i) for a matching edge which after the shrinking could play the role of the petal k in c). In h) only ac could play the role of k. But then cx corresponds to j and it is already present before the shrinking. In i) only bd could play the role of k. But because the augmenting path contains bd, it could not become a petal in the shrinking. The reasoning is exactly the same for case d) and very similar for case g).

Let us finally suppose x is odd.

Odd nodes are made in Steps 3.1, 3.3, 3.5, 3.8a, 3.10a, 3.10b, 3.12a, and 3.13. Let us consider cases c) and d) when x is not the node b (as was designated in Step 5 when v was shrunk). In all the cases in which we make an odd node in the algorithm, the odd node either ends up after a shrinking incident to a matching edge in a petal or can be adjacent to no node, via a matching edge, which is incident with a petal. For example, consider Figure 3.6 c) in Step 3.3. If the augmenting path P_{c_1} or P_{c_2} (where $\ell = c_1c_2$) contains u_1 and w, then u_2 becomes a petal tip and if the augmenting path contains u_2 and w, then u_1 is incident with no petal. If x is the node b, then the augmentation from x passes through x twice, so the triangle is not blocking.

Let us now consider case g). If x was not at some time dummy odd, then x was made odd in Step 3.1, 3.3, or 3.5 and zx is the only nonmatching edge of $\mathcal{G}$ incident with x. Then one of the two nodes adjacent to x via a matching edge must be a petal of v, if x was made odd in 3.1 or 3.3, and there can be no petal k if x was made odd in 3.5. So let us assume x was at some time a dummy odd node. If it became odd in 3.8, 3.10, or 3.12, then there can be no petal k. So let us assume it became odd in 3.13. So there must be two nonmatching edges in $\mathcal{G}$ incident with x. Let us call the other one xz_3. Because z_1 and z_2 are in v, there must have been an edge $m = d_1d_2$ which caused a shrinking such that $z_1 \in P_{d_1}$ and $z_2 \in P_{d_2}$. Exactly one of the two paths P_{d_1} or P_{d_2} contains xz. We have an augmenting path from x of the form $((P^x_{d_1})^{-1}, P_{d_1d_2}, P_{d_2})$. If we replace this with $((P^x_{d_2})^{-1}, P_{d_2d_1}, P_{d_1})$, we do not create the triangle xzw.

Let us finally consider case b). If x is odd, then there are two matching edges: xy_1 and xy_2. As in cases e) and f) both y_1 and y_2 must be contained in v (this is a consequence of observation (3.11)). We can now use the same reasoning that we did for case g) to construct an augmenting path which does not contain zx.

<u>Claim 2</u>: Augmenting on a path of the form (P_1, P_{vw}, P_w) cannot create a triangle due to the combined effect of the three parts.

We have just seen that augmenting on such a path cannot create a triangle which contains an edge in P_{vw}. So the only place a triangle could be created is by the combination of augmenting on P_1 and P_w. Since by our definition of the node b in Step 5, no edge can occur more than once in an augmenting path P, a triangle can occur only if the two paths share a node. Even nodes can occur twice in an augmenting path (see Steps 3.8, 3.10, 3.12) but in none of these cases can the node be passed through twice by different paths. If a node is dummy odd, then it is incident with only two edges of $\mathcal{S}$ and so can be contained in at most one path. So we are left with the case that the common node, say c, is odd. If c was made odd in Steps 3.1, 3.3, 3.5, 3.8, 3.10 or 3.12, then c is incident with at most three edges of $\mathcal{S}$ and so can be contained in at most one augmenting path. So the shared node c can only be an odd node with degree four in $\mathcal{S}$ which was created in Step 3.13. In each augmentation which contains c twice, two matching edges, say z_1x and z_2x, are removed from the matching and two nonmatching edges, say y_1x and y_2x are put into the matching.

In each case, because of the existence of the blocking triangle which contained x when it was dummy odd, there can be no matching edge y_1y_2. For example, consider the constructions in Figures 3.20 and 3.21 in Step 3.13. In no case can there be an edge $vx \in M$ since in every case either v or x is shrunk with a base which is not in the matching. Hence both matching edges adjacent to these bases, which are our candidates for $vx \in M$, are contained in the corresponding shrunk node.

Finally, we observe that, just as in the proof of Theorem 2.1, augmenting on a path of the form (P_1, P_{vw}, P_w) leaves every shrunk node on this path saturated.

Proposition 3.4: The algorithm terminates after a polynomial number of iterations.

Proof: The algorithm only goes to Step 1 after an augmentation (Step 4), which can occur at most $2|V|$ times. Between visits of Step 1, the algorithm constructs an alternating structure by either adding new nodes (Step 3) or by shrinking a set of nodes (Step 5). Each of these possibilities can occur $O(|V|)$ times. So the algorithm terminates after a polynomial number of steps.

Theorem 3.2: When the algorithm terminates, the triangle-free simple 2-matching has maximum cardinality.

Proof: Let M be the matching at the end of the algorithm. Let S be the set of odd nodes, let T be the set of nodes not in $\mathcal{S}$, and let $\mathcal{B}$ be the set of nodes in $V \setminus S \setminus T$ (that is, those nodes which are even or dummy odd).

When the algorithm terminates, the structure $\mathcal{S}$ has the following properties (see Figure 3.27):

1) All odd nodes are saturated and both matching edges from each odd node are incident with even nodes of $\mathcal{S}$.

2) All blossom trees in $\mathcal{B}$ are internally saturated.

3) All edges from even nodes to nodes in T are in the matching.

4) All dummy odd nodes are saturated; one matching edge is in the petal which contains the dummy odd node and the other matching edge is incident with a node in T or another dummy odd node. (Note that, if the two matching edges incident with a dummy odd node are also incident with two even nodes, then we

call the dummy odd node an odd node.)

5) There may be nonmatching edges from dummy odd nodes to nodes in T and to other dummy odd nodes.

6) Dummy odd nodes all occur in triangular petals with one or two matching edges as in Steps 3.2 or 3.4. There is at most one dummy odd node per petal. The second petal tip in any petal which contains a dummy odd node is adjacent to no nodes in T.

7) All nodes in T are saturated.

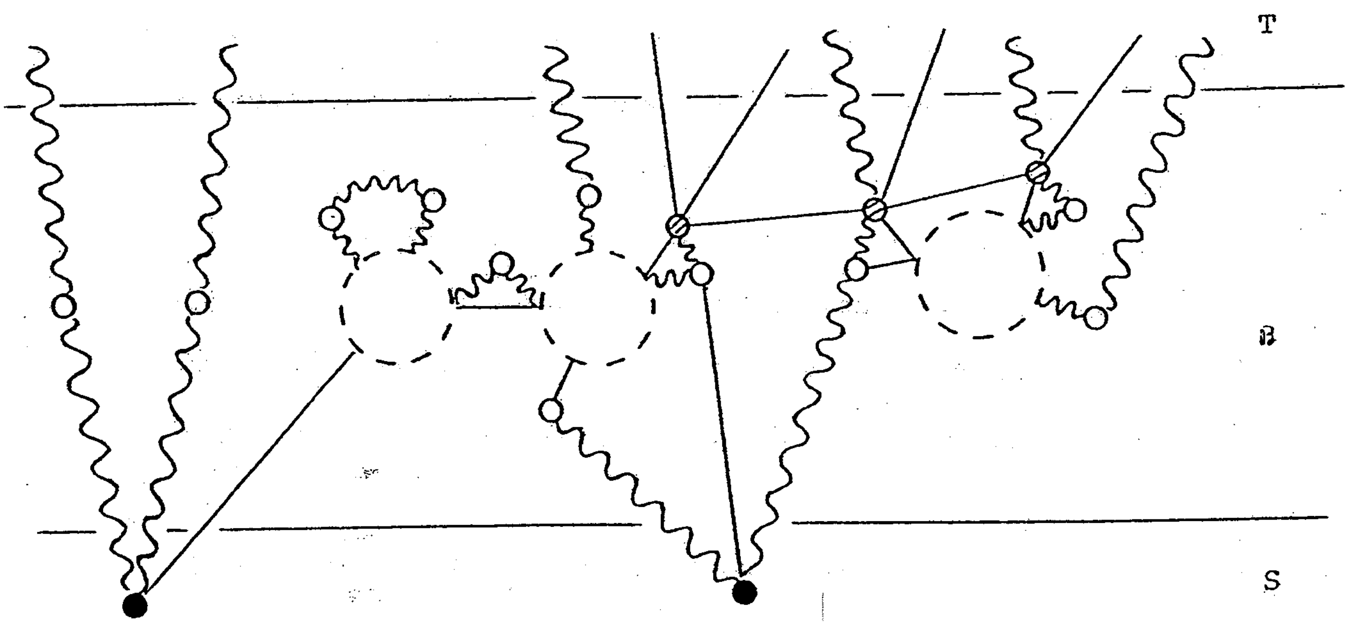

Figure 3.27

Let B be a blossom tree in ℬ which contains dummy odd nodes $v_1,\ldots,v_n$. Note that $B\setminus\{v_1,\ldots,v_n\}$ is a blossom tree by 6). (Let us for now consider a single matching edge to be a degenerate blossom tree in the case that B is a blossom tree consisting of a single triangular petal.) Also note that $B\backslash\{v_1,...,v_n\}$ when internally saturated has the same number of deficiencies which require outside matching edges (i.e. the number of matching edges needed which are not contained in $B\backslash\{v_1,...,v_n\}$) as does B. This is implied by our definition of critical blossom trees since B and $B\backslash\{v_1,...,v_n\}$ have the same number of petals.

So let us consider the blossom trees of $\mathcal{B}$ obtained by putting all the dummy odd nodes into T. Let us call the new sets S, T', and $\mathcal{B}'$.

Consider the new edge petals of $\mathcal{B}'$. We have two edges from T' adjacent to each. Suppose $\overline{M}$ is any matching which uses at most one of these two edges for each petal. Then $\overline{M}$ cannot eliminate more deficiencies in the blossom trees than did M since we have exactly the same number of edges available to use from outside the blossoms as we used in M and each such edge was used in M to eliminate a deficiency (properties 1), 3), and 4)).

So let us consider any matching $\overline{M}$ which uses both edges from T' for some petals of $\mathcal{B}'$. Recall that these pairs of edges are part of the original triangular petals and therefore correspond to a matching which uses no edges from T to this petal (see Property 7) about the second tip in a petal which contains a dummy odd node). Hence we are in effect using fewer edges from outside the blossom trees than we did in M and so cannot eliminate more deficiencies.

Correction

1

Section 5. A Tutte-type Theorem for Triangle-free Simple 2-matchings

In the discussion that follows we will use a slightly different definition of blossom trees: a "blossom tree" is a blossom tree as before except pendent petals of a center may share an attachment node to that center; a node in the center of a "blossom tree" may be a (degenerate) petal; the number of petals incident with a center may be odd or even; and isolated triangles are not considered "blossom trees" (although they are considered pendant triangular petals).

Given $G = (V,E)$, let $S \subseteq V$ and let $A \subseteq V \setminus S$. We obtain the graph $G^A[V \setminus S]$ from $G[V \setminus S]$ by splitting the nodes of A in $G[V \setminus S]$ as follows: For $v \in A$, if the degree of v is 0, do nothing; otherwise, replace v with nodes $v_1, \ldots, v_k$ and replace every edge (u,v) with an edge (u,v_i) so that 1) every node v_i has degree 1 or 2 and after splitting; 2) every degree 2 node is in a triangle; and 3) isolated triangles contain 3 nodes that were split.

With this definition, the components of $G^A[V \setminus S]$ are either isolated nodes, "blossom trees" (without degenerate petals) or isolated triangles. Next, let $B \subseteq V \setminus S \setminus A$ have the following property: if $t \in B$, then there exists a triangle U_t in $G^A[V \setminus S]$ with nodes x, y, z such that the degree of x in $G^A[V \setminus S]$ is 2, the degrees of y and z are > 2, $y = t$ and $z \notin B$. Then, for $t \in B$, we split t into two nodes so that (after all the nodes in B are split) U_t becomes a pendant triangular petal (but, by definition, not an isolated triangle). Both split nodes become tips of petals; in particular, the split node not in U_t is called a (degenerate) petal. Let $T = A \cup B$ and let $G^T[V \setminus S]$ denote the graph resulting from splitting the nodes in T . Finally, we require that 4) if a triangle of $G^T[V \setminus S]$ contains a split node, then each node of the triangle is either split or is a cutnode of the graph. Define

$Q(S,T) = \Sigma(2 - f(t): t \in T) - P(G^T[V\setminus S])$

where

$P(G^T[V\setminus S])$ = number of pendant triangular petals of $G^T[V\setminus S]$; and

$f(t) =$ 0, if the degree of t in $G^T[V\setminus S]$ is 0, or the number of nodes into which t is split, otherwise.

Let $D(S,T)$ = number of odd "blossom trees" of $G^T[V\setminus S]$.

Theorem: G has a perfect triangle-free simple 2-matching iff
$2|S| \geq Q(S,T) + D(S,T)$ for all $S,T \subseteq V$, $S \cap T = \emptyset$ and all allowed splittings of T.

Proof: ($\Rightarrow$) Let M be a perfect triangle-free simple 2-matching of G. Choose any $S,T \subseteq V$, $S \cap T = \emptyset$ and any allowed splitting of T (where $A,B \subseteq V\setminus S$ are as described above). Let $M' = M \cap \gamma(V\setminus S)$. Clearly,

$2|S| \geq 2|V\setminus S| - 2|M'| = 2|V\setminus S| - Q(S,T) - 2|M'| + Q(S,T)$
$= 2|V\setminus S\setminus T| + \Sigma(f(t): t \in T) + P(G^T[V\setminus S]) - 2|M'| + Q(S,T)$

We must show that

(3.12) $2|V\setminus S\setminus T| + \Sigma(f(t): t \in T) + P(G^T[V\setminus S]) - 2|M'| \geq D(S,T)$.

Let us consider the contribution of each component C of $G^T[V \setminus S]$ to the LHS of (3.12). (We abuse notation slightly by saying that nodes of C that have been split are in T.) Thus it suffices to show for each component C of $G^T[V \setminus S]$:

(3.13) $2|(V \setminus S \setminus T) \cap V(C)| + \Sigma(1: t \in T \cap V(C)) + P(C) - 2|M' \cap E(C)|$

≥ 1, if C is an odd "blossom tree," or 0, otherwise.

(3.13) is trivially true if C is an isolated node, since $2|M' \cap E(C)| = 0$. Suppose C is an isolated triangle. By definition, this component is not an odd "blossom tree," so we must show that LHS(3.13) ≥ 0. Since no node of the triangle is a cutnode, every node must be in T; and since the triangle is considered to be a pendant triangular petal, $P(C) = 1$. The result follows.

So, finally, we suppose C is a "blossom tree." Let us define b^C on C as in (3.2), with the addition that degenerate petals v satisfy $b^C(v) = 1$. As before, if each node of C is matched at or below its b^C value by M', then we get that

$$b^C(V(C)) - 2|M' \cap E(C)| \geq 1, \text{ if C is odd, or 0, otherwise.}$$

And since, by definition,

$$2|(V \setminus S \setminus T) \cap V(C)| + \Sigma(1: t \in T \cap V(C)) + P(C) = b^C(V(C)),$$

the result follows. So let us suppose each node of C is not matched at or below its b^C value. Let over(C) = the number of degenerate petals of C matched at value 2 and let under(C) = the number of nodes of C that are i) in B; ii) in triangular petals; iii) matched at value 0; and iv) with other split node matched at value 2. Then

$$b^C(V(C)) - 2|M' \cap E(C)| \geq 1 - over(C) + under(C), \text{ if } C \text{ is odd, and}$$
$$- over(C) + under(C), \text{ otherwise.}$$

Note that if we sum these inequalities over all the components, then the contributions of over() and under() cancel, and the result follows.

($\Leftarrow$) Suppose G = (V,E) does not have a perfect triangle-free simple 2-matching. Apply the cardinality algorithm to G. At termination, let S be the set of odd nodes, and let T = A $\cup$ B, where A is the set of ral even nodes and B is the set of dummy odd nodes. Perform the splitting of nodes in T so that the odd "blossom trees" and isolated triangular petals of the structure **S** are separated from the rest of the graph. One can show (using the properties of **S**) that Q(S,T) + D(S,T) is precisely the number of edges required from S to handle all of the deficiencies among the even nodes (plus the number of odd components formed in V\S\{odd blossoms, isolated nodes and isolated triangles in **S**}). From the algorithm we know that every node in S is saturated and that both matching edges for each node in S are incident with even nodes. Therefore, since this matching is not perfect, 2|S| < Q(S,T) + D(S,T).

Section 5. A Tutte-type Theorem for Triangle-free Simple 2-Matchings

In the discussion which follows we will use a slightly different definition of blossom trees: A "blossom tree" is a blossom tree as before except pendent petals of a center may share an attachment node to that center; the number of petals incident with a center may be odd or even; and isolated triangles are not considered "blossom trees". A "blossom tree" is called odd if it has an odd number of petals.

Given $G = (V,E)$, let $S \subseteq V$, $T \subseteq V \backslash S$, and $U \subseteq T$. We obtain the graph $G^T[V\backslash S]$ from $G[V\backslash S]$ by splitting the nodes of T in $G[V\backslash S]$ as follows: For $v \in T$, replace v with nodes $v_1,\ldots,v_k$ and replace every edge (u,v) with an edge (u,v_i) so that 1) every node v_i has degree 1 or 2; and after splitting: 2) every degree 2 node is in a triangle; 3) isolated triangles contain 3 nodes which were split; and 4) if a triangle contains a split node, then each node of the triangle is either split or is a cutnode of the graph.

With this definition, the components of $G^T[V\backslash S]$ are either "blossom trees" or isolated triangles. Define:

$$Q(S,T,u) = \Sigma(2 - f(u) : u \in U) - P(G^T[V\backslash S])$$

where

$$P(G^T[V\backslash S]) = \text{number of pendent triangular petals of } G^T[V\backslash S]$$

and

$$f(u) = \text{number of nodes into which } u \text{ is split.}$$

Let $D(S,T,U)$ = number of odd "blossom trees" of $G^T[V\backslash S]$.

<u>Theorem 3.3</u>: G has a perfect triangle-free simple 2-matching iff $2|S| \geq Q(S,T,U) + D(S,T,U)$ for all S,T,U.

<u>Proof</u>: (⇒) Let M be a perfect triangle-free simple 2-matching of G. Choose any S,T,U and splitting as described above. Let $M' = M \cap \gamma(V\backslash S)$. Clearly,

$$\begin{aligned} 2|S| &\geq 2|V\backslash S| - 2|M'| = 2|V\backslash S| - Q(S,T,U) - 2|M'| + Q(S,T,U) \\ &= 2|V\backslash S\backslash U| + \Sigma(f(u) : u \in U) + P(G^T[V\backslash S]) - 2|M'| + Q(S,T,U). \end{aligned}$$

We must show that

$$(3.12) \quad 2|V\backslash S\backslash U| + \Sigma(f(u) : u \in U) + P(G^T[V\backslash S]) - 2|M'| \geq D(S,T,U).$$

Let us consider the contribution of each component C of $G^T[V\backslash S]$ to the LHS of (3.12). We want to show that

$$(3.13) \quad 2|(V\backslash S\backslash U) \cap V(C)| + \Sigma(1 : u \in U \cap V(C)) + P(C) - 2|M' \cap E(C)| \geq \begin{cases} 1 & \text{if } C \text{ is an odd "blossom tree"} \\ 0 & \text{otherwise} \end{cases}$$

(3.13) is trivially true if C is an isolated node since

$$2|M' \cap E(C)| = 0.$$

Suppose C is an isolated triangle. By definition, this component is not odd, so we must show that LHS(3.13) ≥ 0. Since no node of the triangle is a cutnode, every node must be in T. The LHS(3.13) is minimized if, in addition, all 3 nodes are in u and the triangle is saturated by M' in which case LHS(3.13) $= 0$. (Such a triangle does count as a pendent triangular petal.)

So, finally, let us suppose C is a nontrivial "blossom tree". If we define b^C on the "blossom tree" C as in (3.2) we get

$$b^C(V(C)) - 2|M' \cap E(C)| \geq \begin{cases} 1 & \text{if } C \text{ is an odd "blossom tree"} \\ 0 & \text{otherwise.} \end{cases}$$

So all we need to show is

$$(3.14) \qquad 2|(V\backslash S\backslash U) \cap V(C)| + \Sigma(1 : u \in U \cap V(C)) + P(C) \geq b^C(V(C)).$$

All nodes in centers contribute 2 to the LHS (3.14) as well as to the RHS (3.14). Tips of edge petals and nonpendent triangular petals each contribute at least 1 to the LHS (3.14) (i.e. if they are in U) and exactly 1 to the RHS (3.14). The two tips of a pendent triangular petal contribute at least 3 to the LHS (3.14) (i.e. if they are both in U) and exactly 3 to the RHS (3.14). Hence (3.14) holds and we are done.

(⇐) Suppose $G = (V,E)$ does not have a perfect triangle-free simple 2-matching. Apply the cardinality algorithm to G. At termination, let S be the set of odd nodes, let T be the set of real nodes of S in $V\backslash S$ which have degree ≥ 1, and let u be the set of real even nodes of T. Perform the splitting of nodes in T so that the "blossom trees" and isolated triangular petals are separated from the rest of the graph. Note that since our "blossom trees" and isolated triangular petals are critical, they individually need one edge from outside for every petal, plus 1 (this is exactly what (3.1) states). However, the total number of edges needed, for all of these structures together, is reduced by $k - 1$ for every tip which is shared by k blossom trees. Using this fact together with properties 3) and 4) from the proof of Theorem 3.2 it is easy to see that $Q(S,T,U) + D(S,T,U)$ is precisely the number of edges required from S to handle all of the deficiencies among the even nodes. From the algorithm we know that every node in S is saturated and both matching edges for each odd node are incident with even nodes. Therefore, since this matching is not perfect,

$$2|S| < Q(S,T,U) + D(S,T,U).$$

Section 6. A Conjecture About the Weighted Triangle-Free Simple 2-Matching Problem

For any graph G, let $P(G)$ denote the convex hull of the incidence vectors of triangle-free simple 2-matchings of G. In this section we conjecture a complete polyhedral characterization of $P(G)$. In order to do this we must introduce another class of valid inequalities for $P(G)$; that is, the inequalities associated with the simple blossom trees with edge or triangular petals are not sufficient.

In Section 1 of this chapter we defined "simple blossom trees" and "simple blossom trees with edge or triangular petals". Let us now consider "blossom trees", in general. Just as the simple blossom trees are an extension of the "clique trees" of Grötschel and Pulleyblank [81], so are blossom trees. (Note that, given a graph $G = (V,E)$, an articulation set $S \subset V$ is a subset of nodes such that $G[V \setminus S]$ has more components than G.)

A blossom tree is a connected graph B such that

(i) it contains at least three nodes,

(ii) centers and petals are connected node induced subgraphs,

(iii) the subgraphs induced by the nodes in each petal have no cutnodes,

(iv) a petal and a center can have at most two common nodes and these common nodes must be articulation sets of B,

(v) no two centers have a common node,

(vi) no two petals have a common node,

(vii) each center is adjacent to an odd number of petals,

(viii) each petal contains at least one node which belongs to no center.

For the triangle-free simple 2-matching problem, the relevant structures are the blossom trees such that

(ix) each petal contains two or three nodes.

Because triangular petals may now share two nodes with a center, these structures clearly generalize the simple blossom trees with edge or triangular petals. (See Figure 3.28.) Note that (iii)' implies that if a petal and center have two common nodes u and v, then the edge (u,v) occurs in both the petal and center. A graph which satisfies properties (i) - (ix) is called a <u>blossom tree with edge or triangular petals</u>. For simplicity these graphs will be referred to as "blossom trees" in this section.

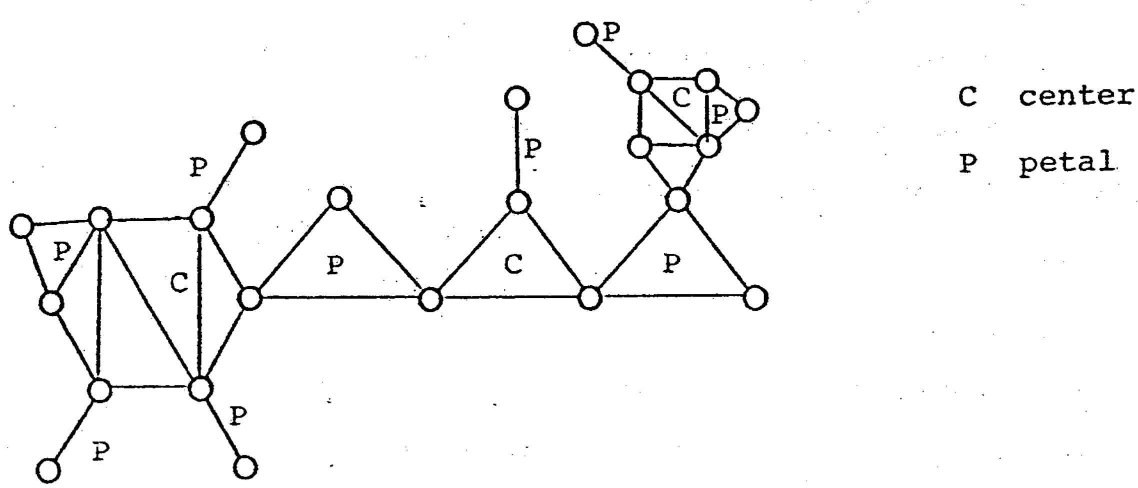

Figure 3.28. A Blossom Tree

Suppose we have a blossom tree B with centers $C_1,\ldots,C_r$ and petals $P_1,\ldots,P_s$. Then we associate with B the following inequality.

$$\sum_{i=1}^{r} x(E(C_i)) + \sum_{j=1}^{s} x(E(P_j)) \leq \sum_{i=1}^{r} |V(C_i)| + \sum_{j=1}^{s} (|V(P_j)| - p_j) - \frac{s+1}{2}$$

where for every petal P_j, $p_j \in \{0,1,2\}$ and denotes the number of handles which intersect P_j, $j = 1,\ldots,s$. Note that if a center and petal share an edge, then the inequality is 0 - 1 - 2. When B is simple, the inequality reduces to (2.1) since

$$\sum_{i=1}^{r} |V(C_i)| + \sum_{j=1}^{s} (|V(P_j)| - p_j),$$

in this case, equals n, the number of nodes in B.

Grötschel and Pulleyblank [81] prove that the clique tree inequalities are facets for the traveling salesman problem. Their proof that the clique tree inequalities are valid for the traveling salesman problem can easily be modified to prove that the blossom tree inequalities are valid for the triangle-free simple 2-matching problem. The sets of clique tree inequalities and blossom tree inequalities have nonempty intersection (that is, when the centers of the blossom trees are cliques).

Therefore, since the triangle-free simple 2-matching problem is a relaxation of the traveling salesman problem, we must include blossom tree inequalities in a complete polyhedral characterization of P(G). We conjecture that these inequalities are sufficient.

<u>Conjecture</u>: For any graph G, $P(G)$ is characterized by the following inequalities:

$$0 \leq x_e \leq 1 \quad \text{for all } e \in E$$
$$x(\delta(v)) \leq 2 \quad \text{for all } v \in V$$
$$ax \leq \alpha \quad \text{for all blossom trees of } G.$$

<u>Proposed proof</u>: Using an Edmonds' style primal-dual algorithm for the weighted triangle-free simple 2-matching problem. In particular, we use the cardinality algorithm of Section 2 in the primal part of the algorithm.

We next give a brief and informal example of where the more general class of blossom trees is needed in an algorithm for the weighted problem.

Suppose we have set up a weighted algorithm for this problem exactly as we did for the simple 2-matching problem in the last chapter. Also suppose that at some point in an application of the algorithm we have grown the structure illustrated in Figure 3.29 and are considering the edge j in the edge selection step.

According to our cardinality algorithm, we should shrink to obtain a simple blossom tree in which (x_1,x_2) is an edge petal. As defined in the proof of Theorem 3.1, we should then set $P_{x_1} = (x_1,\ldots,x_6,r)$. However, due to a previous dual change we may have $z_e > 0$. If this is the case, then we cannot augment along P_{x_1} due to the complementary slackness condition which states $z_e > 0 \Rightarrow x(e) = 1$ where x is the current matching. So we must lower z_e to zero before identifying this simple blossom tree. To do this we identify the blossom tree in which

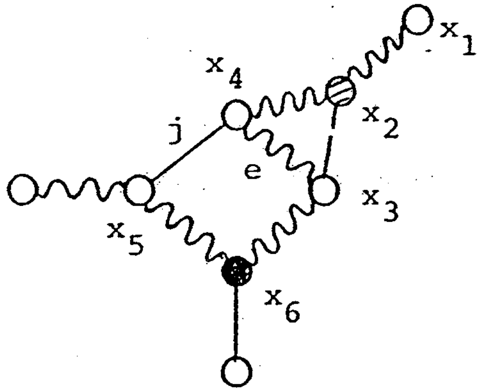

Figure 3.29

x_2, x_3, x_4 is a triangular petal sharing two nodes with the center x_3, x_4, x_5, x_6. Hence the edge e has a coefficient of 2 associated with it in this inequality.

When we raise the dual variable associated with this blossom tree we must lower the dual variable z_e by the same amount in order to keep e in the equality subgraph. If, after some dual changes, $z_e = 0$, then we may add the node x_1 as a petal tip of the simple blossom tree, as we originally intended.

Chapter 4

A GENERALIZATION OF MATCHING THEORY

Section 0. Introduction

In this chapter we extend many results of matching theory to what we call hypomatchings. For convenience, let us briefly restate some definitions given in Chapter 1 and also give some new ones.

Given a graph $G = (V,E)$ and a family F of subsets of V, an F-packing is a subfamily $J \subseteq F$ such that every node of G belongs to at most one member of J. Let us say a subset of nodes is hypomatchable if it induces a hypomatchable subgraph of G. Then, when H denotes a family of hypomatchable node sets and $F = E \cup H$, F-packings are called hypomatchings. Given a hypomatching J, any node which belongs to one member of J is said to be covered by J. A maximum hypomatching is one which covers the maximum number of nodes of G. A perfect hypomatching of a subgraph $G[S]$ is a hypomatching of G which covers all the nodes of S but no other nodes. The graph $G[S]$ is said to be critical relative to F if it does not have a perfect hypomatching but, for every $j \in S$, the graph $G[S \setminus \{j\}]$ has one. (See Cornuejols, Hartvigsen, and Pulleyblank [82], and Cornuejols and Hartvigsen [83].)

Section 1. A Relationship Between Maximum Matchings and Maximum Hypomatchings

We first generalize a result known as the Gallai-Edmonds Theorem (see Edmonds [65a]).

Given a graph G, consider the following partition of its nodes into three sets O, I, R.

(i) A node of G belongs to O if and only if it is not matched in at least one maximum matching;

(ii) I is the set of nodes of G which are matched in every maximum matching and are adjacent to at least one node of O;

(iii) R is the set of nodes of G which are matched in every maximum matching but are not adjacent to any node of O.

The Gallai-Edmonds theorem states that

(a) every component of G[O] is hypomatchable;

(b) a matching of G is a maximum matching if and only if

(iv) the nodes of R are matched among themselves;

(v) in each component of G[O], all but one of the nodes are matched among themselves;

(vi) each node of I is matched to a node in a distinct component of G[O].

The partition O, I, R can be obtained by applying Edmond's matching algorithm. Consider the alternating forest at termination of the algorithm. The set of nodes which are either outer nodes of the forest or inside shrunk outer nodes is the set O. The set of inner nodes of the forest forms the set I. The rest of the nodes of G is R. (The letters O, I, and R stand for outer, inner, and remaining nodes, respectively.)

Now we turn to maximum hypomatchings. Recall that $F = E(G) \cup H$ where every $S \in H$ is a hypomatchable subset of the nodes of G. Consider the following partition of the nodes of G into three sets O(F), I(F), and R(F).

(i') a node of G belongs to O(F) if and only if it is not covered in at least one maximum hypomatching;

(ii') I(F) is the set of nodes of G which are covered in every maximum hypomatching and are adjacent to at least one node of O(F);

(iii') R(F) is the set of nodes of G which are covered in every maximum hypomatching but are not adjacent to any node of O(F).

Given a hypomatching J of $G = (V,E)$, a node of $S \subseteq V$ is said to be <u>internally covered</u> in S if it is covered by a member T of J such that $T \subseteq S$.

<u>Theorem 4.1</u>: The partition O(F), I(F), R(F) is such that

(a') every component of G[O(F)] is critical;

(b') a hypomatching of G is maximum if and only if (iv') all the nodes of $R(F)$ are internally covered in $R(F)$;

(v') in each component of $G[O(F)]$, all but one of the nodes are internally covered in the component;

(vi') each node of $I(F)$ is matched to a node in a distinct component of $G[O(F)]$.

In order to prove Theorem 4.1 we need the following lemma.

Lemma 4.2: Let S and T be two subsets of the nodes of $G = (V,E)$ and assume that T is hypomatchable. If S and T have p (≥ 1) common nodes, then at most $p-1$ critical connected components of $G[V-S]$ have a node set C such that $G[C-T]$ admits a perfect hypomatching.

Proof of Lemma 4.2: Let $\hat{M}$ be a near-perfect matching of $G[T]$ leaving one node of S unmatched. Now let C be the node set of any critical connected component of $G[V-S]$ such that the nodes of $C \cap T$ are matched among themselves by the matching $\hat{M}$. If $G[C-T]$ had a perfect hypomatching, then completing it with the edges of $\hat{M}$ in $G[C \cap T]$ would produce a perfect hypomatching of $G[C]$, a contradiction to the fact that $G[C]$ is critical. So the critical components of $G[V-S]$ such that $G[C-T]$ has a perfect matching must have at least one of their nodes matched with a node of S in the matching $\hat{M}$. There are at most $p-1$ such components since $\hat{M}$ leaves one node of S unmatched.

Now we prove the theorem.

Proof of Theorem 4.1: Consider the sets O, I, and R defined by (i)-(iii). Let $\overline{G}$ be the bipartite graph obtained from $G[O \cup I]$ by shrinking each connected component of $G[O]$ to a single node and by removing all the edges of $G[I]$. If a component of $G[O]$ is critical, the corresponding node of $\overline{G}$ will also be called critical. As a consequence of statement (vi) of the Gallai-Edmonds theorem, every maximum matching of $\overline{G}$ matches all the nodes of I. Let $\overline{M}$ be such a maximum matching of $\overline{G}$ with the property that the number of critical matched nodes is the largest possible among all maximum matchings of $\overline{G}$.

If every critical node of $\overline{G}$ is matched by $\overline{M}$, set R(F) to be the node set of G and $O(F) = I(F) = \emptyset$. Otherwise, let the critical unmatched nodes of $\overline{G}$ be defined as the roots of the trees of a forest A. These nodes will also be called outer nodes of A. If some edge e joins an outer node of A to a node $i \in I$ not in A, let $m = (i,j)$ be the edge of $\overline{M}$ incident with i. Grow the forest A by adding to A the edges e and m and call the nodes i and j inner and outer nodes of A, respectively. (Note that the node j must be critical, otherwise by interchanging the edges in and out of $\overline{M}$ on the path of A from j to the root, one more critical node could be matched, contradicting the assumption about $\overline{M}$.) Keep growing the forest A as described above until every edge incident with an outer node of A is also incident with an inner node of A.

Then let I(F) be the set of inner nodes of A, O(F) the set of nodes of G(O) contained in outer nodes of A, and R(F) the remaining nodes of G. So $I(F) \subseteq I$, $O(F) \subseteq O$ and $R(F) \supseteq R$. Note also that, by construction of A, every component of G[O(F)] is critical and no edge of G joins O(F) to R(F). We will show that the partition O(F), I(F), R(F) just constructed is in fact the unique partition defined by (i')-(iii').

Before doing this, we exhibit a hypomatching J of G which leaves s uncovered nodes, where s is defined to be the number of components of G[O(F)] minus the cardinality of I(F). We define J separately on G[R], $\bar{G}$ and G[O]. In G[R], take J to be any perfect matching (this is possible by statement (iii) of the Gallai-Edmonds theorem). In $\bar{G}$, take J to be identical to $\bar{M}$. Finally, in G[O], take J to be a hypomatching which internally covers all the nodes of the noncritical components incident with no edge of $\bar{M}$, and all but one of the nodes of the remaining components of G[O]. (When such a component contains a node u incident with $\bar{M}$, u is the only node of the component which is not internally covered.) Since $\bar{M}$ matches every node of I(F) with a node of O(F), so does J, leaving only s uncovered nodes in O(F). Every node of R(F) is covered by J, so we have the announced hypomatching.

In fact, the hypomatching J just constructed is maximum: A consequence of Lemma 4.2 with $S \equiv I(F)$ is that any hypomatching of G which does not cover all the nodes of I(F) or which contains a hypomatchable set $T \in H$ with at least one node in $I(F)$, must leave more than s uncovered nodes in $O(F)$. By matching all the nodes of $I(F)$ to nodes of $O(F)$ at least s nodes of $O(F)$ must remain uncovered, and in fact it is possible to leave exactly s uncovered nodes in G, as shown by the hypomatching J constructed earlier. This shows that J is a maximum hypomatching.

This also proves that every maximum hypomatching of G satisfies (iv'), (v'), and (vi'). Conversely any hypomatching which satisfies (iv'), (v') and (vi') leaves only s uncovered nodes and therefore is maximum.

The fact that maximum hypomatchings satisfy (iv') and (vi') implies (ii') and (iii'). So only statement (i') remains to be proved. Consider the forest A in $\bar{G}$. Any critical outer node j of A can be left unmatched by some matching $\tilde{M}$ which has the same cardinality as $\bar{M}$ and leaves unmatched the same non-critical nodes as $\bar{M}$. Specifically, if j is a critical node of A matched by $\bar{M}$, then construct $\tilde{M}$ from $\bar{M}$ by interchanging the edges in and out of $\bar{M}$ on the path of A from j to a root of A. Now the matching $\tilde{M}$ can be used instead of $\bar{M}$ to construct a maximum hypomatching J as done earlier. Furthermore, in the critical component of $G[O(F)]$ left unmatched by $\tilde{M}$, any node can be left uncovered. This proves statement (i') and completes the proof of Theorem 4.1. $\blacksquare$

This structural theorem has many consequences, as we shall see in the next four theorems.

Theorem 4.3: Consider a graph G and two families $F_1 = E(G) \cup H_1$ and $F_2 = E(G) \cup H_2$ such that the node sets in H_1 and H_2 are hypomatchable. If $H_1 \subseteq H_2$, then the partitions $O(F_i)$, $I(F_i)$, $R(F_i)$, $i = 1,2$, satisfy $O(F_2) \subseteq O(F_1)$, $I(F_2) \subseteq I(F_1)$ [and therefore $R(F_2) \supseteq R(F_1)$].

Proof: The property $O(F_2) \subseteq O(F_1)$ follows from (i') and the fact that every hypomatching relative to F_2 is also a hypomatching relative to F_1.

The property $I(F_2) \subseteq I(F_1)$ follows from $O(F_2) \subseteq O(F_1)$ and the fact that $I(F_i)$ is exactly the set of nodes of G adjacent to $O(F_i)$, $i = 1,2$ (see (ii') and (iii')). $\blacksquare$

The next result generalizes Theorem 1.9 of Tutte [47].

Theorem 4.4: A graph $G = (V,E)$ has a perfect hypomatching if and only if, for every $S \subseteq V$, the graph $G[V-S]$ contains at most $|S|$ critical connected components.

Proof: If G does not have a perfect hypomatching, then $O(F) \neq \emptyset$ in Theorem 4.1. Let $S = I(F)$. By Theorem 4.1, the number of critical components in $G[O(F)]$ is larger than $|S|$ (since a maximum hypomatching matches each node of S with a node in a different component of $G[O(F)]$ and still leaves at least one component of $G[O(F)]$ unmatched). By (iii') the critical components in $G[O(F)]$ remain critical in $G[O(F) \cup R(F)] \equiv G[V-S]$.

Conversely, assume that G has a perfect hypomatching J. Consider any $S \subseteq V$. If no hypomatchable set $T \in J$ contains a node of S, then every critical component of $G[V-S]$ has to be matched to some node of S by an edge of J, proving the theorem. Otherwise, $|(T \cap S)| = p \geq 1$ for some $T \in J$. Then by Lemma 4.2 the number of critical components of $G[V-S]$ having a node set C such that $G[C-T]$ admits a perfect hypomatching is at most $p-1$. The theorem follows by induction on the number of hypomatchable sets of J which intersect S. ∎

Section 2. Maximum Hypomatchings with a Minimum Number of Hypomatchable Sets

The next result generalizes a theorem of Urhy [75] relating maximum matchings and fractional matchings. Let $G = (V,E)$ be a graph and let $F = E \cup H$ where H is a family of hypomatchable sets.

Theorem 4.5: Let $J \subseteq F$ be a maximum hypomatching containing a minimum number of hypomatchable sets. Then the matching obtained by taking the edges of J and near-perfectly matching the hypomatchable sets of J, is a maximum matching.

Our proof of Theorem 4.5 uses the following lemma.

Lemma 4.6: A noncritical hypomatchable subgraph $G[S]$ of G has a perfect hypomatching using only one of the hypomatchable sets in H.

Proof: Let P be a perfect hypomatching of $G[S]$ containing a minimum number of hypomatchable graphs from F. Suppose that P contains more than one such hypomatchable subgraph. The number of these subgraphs is odd, and hence at least three, since $G[S]$ has an odd number of vertices. We define a matching M of $G[S]$ as follows. M contains all the edges of P and comprises a near-perfect matching of each of the hypomatchable subgraphs in P.

Then every hypomatchable subgraph H of P will contain exactly one node $v(H)$ not covered by M, and these are the only nodes not covered by M. If we delete from $G[S]$ one such node $v(H)$, the resulting graph has a perfect matching, that is M is not maximum. So by Berge's theorem 1.7 there exists an augmenting path relative to M, joining $v(H_1)$ to $v(H_2)$ where H_1 and H_2 are some hypomatchable subgraphs in P. This augmenting path must contain a path D whose endnodes belong to different hypomatchable graphs of P, say H_1' and H_2', and such that no other node of D belongs to a hypomatchable graph of P. Note that the first and last edges of D are not in the hypomatching P, so D is an augmenting path for P. By changing P in the hypomatchable graphs H_1' and H_2' and by applying the augmentation along D, we can obtain a new perfect hypomatching with a smaller number of hypomatchable subgraphs. ■

Proof of Theorem 4.5: Let O, I, R be the node sets defined in the Gallai-Edmonds theorem and $O(F)$, $I(F)$, $R(F)$ those defined in Theorem 4.1. Set $I - I(F) = L$ and $O - O(F) = \Omega$.

Consider J as defined in Theorem 4.5. By Theorem 4.1 every node of $I(F)$ is matched by an edge of J to a node of $O(F)$ and in every connected component of $G[O(F)]$ all but one of the nodes are internally covered. Only edges are needed in these near-perfect hypomatchings since the components of $G[O(F)]$ are hypomatchable. So the nodes of $O(F) \cup I(F)$ are only matched by edges of J.

The nodes of $R(F) = R \cup \Omega \cup L$ are internally covered. Since the nodes of Ω are only joined to L in $G[R(F)]$, the number of hypomatchable subgraphs needed to internally cover the nodes of $R(F)$ is at least equal to the number of components of Ω minus the cardinality of L. In fact, the matching $\bar{M}$ defined in the proof of Theorem 4.1 shows that no more are needed since
(1) the nodes of R are perfectly matched among themselves,
(2) every node of L is matched to a component of Ω and
(3) in each component of Ω which is not matched to L, all the nodes can be internally covered using only one hypomatchable set by Lemma 4.6. This completes the proof of Theorem 4.5. ■

A set of nodes S is separable if and only if there exists a maximum matching which does not use any edge with exactly one end in S. The next result generalizes a theorem of Balas [81].

Theorem 4.7: Given $G = (V,E)$ and $F = E \cup H$, a maximum matching is also a maximum hypomatching of G if and only if none of the hypomatchable sets in H is separable.

Proof: The necessity follows from the observation that, if some hypomatchable set $S \in H$ were separable, then a maximum matching M using no edge in the boundary of S would leave one node of S unmatched, but a hypomatching identical to M on $G[V-S]$ and using S would cover one more node of G.

Conversely, suppose G does not have a maximum hypomatching using just edges. Consider one which uses a minimum number of hypomatchable sets of H. By Theorem 4.5 these sets are separable. This completes the proof. ∎

Section 3. Hypomatching Matroid

Let $G = (V,E)$ and $F = E \cup H$ where H is a family of hypomatchable sets. A node set $S \subseteq V$ is said to be independent if there exists a hypomatching $J \subseteq F$, such that S is a subset of the nodes covered by J. Let M be the family of all independent sets. The system (V,M) is an independence system, i.e., $S \in M$ and $T \subseteq S \Rightarrow T \in M$. When a hypomatching covers all the nodes of a set S, we say that it covers S.

Theorem 4.8: The independence system (V,M) is a matroid.

Proof: Consider $G[O(F) \cup I(F)]$. In this graph, we say that a node set $X \in M'$ if and only if X can be covered by a matching