Fingering Systems for Electronic Musical Instruments

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Abstract

Consider an electronic musical instrument that plays sequences of notes, one at a time, from the Western chromatic scale. The instrument has a set of keys and pressing each combination of keys with the fingers corresponds to a note. In this paper we consider the question: What mapping from combinations of keys to notes results in an instrument that is optimally easy to play? We operationalize the notion "easy to play" by looking for mappings that (1) require a small number of simple finger movements when playing various common structured sequences of notes (such as scales and arpeggios) as well as samples of melodies from Western music; and (2) are conceptually simple, where the keys are partitioned into two groups so that one group selects the octave and the other group selects the pitch class.

Keywords: musical instrument design; Gray codes; combinatorial optimization

AMS codes: 90C27, 05A99

1 Introduction

In this paper we consider musical instruments that play sequences of notes, one at a time, from the Western chromatic scale. In particular, we consider instruments that have a set of keys (or locations) such that pressing any combination of them with the fingers results in an individual note. For example, an instrument could have 15 keys and pressing keys 1, 2, 4, and 7 would allow the musician to produce middle C. We are interested in finding assignments (or mappings) of key combinations to notes that yield instruments that are easy to play. (The method of producing the sound – reed vibration, lip vibration, string bowing, and so on – does not concern us.) In this introduction we discuss some related existing instruments, the type of instruments we design, and what we mean by "easy to play." We also introduce our results, discuss related work, and give an outline of the paper.

This work extends previous work by the author in [13]. Although the background material in this introduction is closely based on the introduction in [13], most of the results in this paper are novel. The contributions of this paper and how it differs from [13] are discussed at the end of this section.

Many traditional, or acoustic, instruments are of the type just described. The woodwinds are perhaps the best examples, but the orchestral strings, guitars, and keyboards are also examples. (Although these latter instruments can also play more than one note at a time, we are not concerned with this feature in this paper.) We point out that for traditional instruments, the possible mappings from combinations of keys to notes are greatly limited by physics. However, for electronic instruments of this type, such limitations are largely relaxed. Hence our purpose in this paper is to focus on electronic instruments and to exploit this freedom.

Examples of electronic instruments of this type include the Electric Wind Instrument (EWI) by Akai, the Yamaha WX, the Morrison Digital Trumpet (by Steve Marshall and James Morrison), the digital flute (by Yunik, Borys, and Swift; see [31]), the Bleauregard (by Gerald Beauregard; see [1]), the MIDI horn (by John Talbert; see [5]), the Hirn (by Perry Cook; see [6]), and the Pipe (by Gary Scavone; see [26]). The book [19] by Miranda and Wanderley discusses the last five of these instruments in detail. (These electronic instruments work by sending MIDI signals to a synthesizer followed by a speaker.) Many other "toy" instruments of this sort have been designed for handheld devices such as the iPhone; examples are WIVI Band^{TM} by Wallander Instruments and the Ocarina^{TM} by Smule. With the exception of the digital flute, the Bleauregard, and the Ocarina^{TM}, the way these electronic instruments assign combinations of keys to notes is based on the traditional examples.

We next informally introduce the type of instrument we study by considering an example. The instrument in our example has seven keys, labeled $1, 2, \ldots, 7$, and can play eight consecutive 12-note octaves, whose notes are labeled $C_1, C\#_1, D_1, D\#_1, \ldots, B_1, \ldots, C_8, \ldots, B_8$. (In this standard music notation, the subsripts denote the octave and the other symbols represent pitch classes, that is, notes that are separated by an integral number of octaves.) The mapping from the key combinations (more often called *fingerings*) to the notes in the first two octaves is shown in Figure 1. Consider the two matrices of 0s and 1s in the rectangles: The rows are labeled by the notes and the columns are labeled by the keys. The 1s in each row indicate, by their column labels, a subset of keys (called a *primary fingering*) that the musician presses to play the note labeling that row. For example, E_1 is played by pressing the keys 4 and 5. Observe that some notes, such as B_1, have a second string of 0s and 1s to the right, outside the rectangle, indicating a second subset of keys (called an *alternate fingering*) that describe an alternate way to play the note labeling that row. For example, B_1 can be played by pressing the keys 6 and 7 or by pressing the keys 1, 5, and 7. (Alternate fingerings are common with woodwinds where they help make certain passages easier to play. Our alternate fingerings play a

	1234567			1234567	
C_1	0000000		C_2	1000000	0000110
$C\#_1$	0000001		$C\#_2$	1000001	0000111
D_1	0001000		D_2	1001000	
$D\#_1$	0001001		$D\#_2$	1001001	
E_1	0001100		E_2	1001100	
F_1	0001101		F_2	1001101	
$F\#_1$	0001110		$F\#_2$	1001110	
G_1	0001111		G_2	1001111	
$G\#_1$	0001010		$G\#_2$	1001010	
A_1	0001011		A_2	1001011	
$A\#_1$	0000010	1000100	$A\#_2$	1000010	1100100
B_1	0000011	1000101	B_2	1000011	1100101

Figure 1: Example of the first two octaves of a fingering system.

similar role.)

The keys for this instrument could be arranged, for example, in a row on a cylinder resembling a traditional woodwind instrument (such as the Akai EWI and Yamaha WX) or on a touch sensitive computer screen. Because the instrument has only seven keys, a musician can dedicate one finger to each key and therefore sequences of notes can be played with combinations of simple up and down finger movements. We contend that this should contribute to ease of play. Contrast this, for example, with the situation for stringed instruments and keyboards, which typically have many more keys or locations than fingers. For these instruments a musician must contend, for each note played, with finding the key or location to be pressed as well as deciding which finger to use. (Note, however, that keyboards have a different ease-of-play advantage: the mapping from keys to notes is simpler than our proposed instruments in the sense that it is one-to-one with a left-to-right, low-pitch-to-high-pitch orientation.)

The instrument in Figure 1 has an additional property (not completely evident in the figure since only two octaves are displayed): Two notes have the same primary fingerings on keys $1, 2, 3$ if and only if they are in the same octave; and two notes have the same primary fingerings on keys $4, 5, 6, 7$ if and only if they are in the same pitch class. Such instruments are called *partitionable*. For example, in Figure 1, the primary fingerings for A_1 and B_1 agree on keys $1, 2, 3$ and the primary fingerings for B_1 and B_2 agree on keys $4, 5, 6, 7$. This means that a musician needs to learn only eight combinations on the first three keys and 12 combinations on the last four keys in order to play all the notes in the instrument's 96 note range (using primary fingerings). Furthermore, this classification of the notes by octaves and pitch classes is conceptually quite natural for musicians. We believe this partitionable property should also contribute to making our instruments easy to play. (A somewhat different design of the octave keys can make this type of instrument even easier to play; see Remark 1.)

All instruments we present in this paper have the form of the above example: They are partitionable with seven keys and a range of eight octaves. We limit our consideration to the seven-key, eight-octave model to make our definitions and analysis easier to present, and because it yields a practical instrument. However, the same approach can be used for instruments with any range size and corresponding number of keys. The results would be essentially identical.

In order to compare different mappings we select several collections of sequences of notes that, in whole or part, occur commonly in Western music. First, we consider various scales and arpeggios over the range of the instrument (e.g., see [22], [17], and [25]). Virtually all introductory books, for learning to play the type of instrument considered in this paper, stress the importance of learning to play scales and arpeggios. In addition, we consider collections containing all intervals of a fixed size over the range of the instrument. Finally, we consider two collections of melodies, one is a sample mostly from well-known European composers and the other is a sample from folk music. These samples appeared in a paper by Vos and Troost [28], in a different context. We compare mappings by measuring how difficult it is to play these collections using the mappings. Since each key has a dedicated finger, we use a function that assigns a cost based on the number of fingers that move up or down, pressing or releasing a key, in playing one note after another in a sequence of notes. In the simplest case, this function would simply count the number of finger movements. For example, the difficulty (or cost) of playing the sequence $A\#_1, C_2, D_2$ on the instrument in Figure 1, using primary fingerings, would be 3, since keys 1 and 6 change in playing $A\#_1$ then C_2, and key 4 changes in playing C_2 then D_2. Observe that if we use the alternate fingering for $A\#_1$, then the cost of playing this sequence would drop to 2. The cost of playing a collection of sequences with primary fingerings is simply the sum of the costs of playing each sequence in the collection using the primary fingerings. The cost of playing a collection of sequences with *free* fingerings is the best total cost that can be achieved where both primary and alternate fingerings are allowed on each sequence. (Hence finding the cost of playing a sequence of notes using free fingerings is an optimization problem.) In general, we seek to identify instruments that have low cost when playing common sequences of notes. We contend that this too contributes to making an instrument easy to play. We also consider some more general cost functions for measuring the difficulty of playing certain sequences of notes. We see that many of our main results continue to hold if we merely assume that the cost function increases with the number of finger movements.

Observe that with seven keys there are $2^7 = 128$ different fingerings, which is the minimum number of keys required to play the 96 notes in eight octaves. Other instrument designs with more keys can also satisfy our ease-of-play criteria. We comment on one possibility in Remark 1. However, we show, perhaps a bit surprisingly, that, in terms of counting simple finger movements, there is no advantage to using more than seven keys for the primary fingerings, in a number of cases.

There are a huge number of different partitionable instruments. In fact, there are approximately 8.7×10^{11} different mappings for the primary fingerings on the keys $4, 5, 6, 7$. (See Section 7 for the easy calculation.) Hence enumerating the mappings to find the best for a particular collection of note sequences is not possible. Another approach to finding optimal mappings is to use the techniques of optimization, which have been applied to similar problems (e.g., the assignment problem; see [2]). However, these techniques do not seem applicable in this case, partly due to the unique form of the cost measure and the existence of alternate fingerings. So we have chosen a different strategy. We primarily limit our search for low cost instruments to a special class of partitionable instruments called *basic*. (The basic instruments are a subclass of what we call *interval-based* instruments. These classes are defined in Section 2.) We show that this class has some nice properties. Roughly speaking, for a basic instrument, the keys $4, 5, 6, 7$ are partitioned into subsets, so that each subset tends to control movement between notes in interval jumps of a fixed size. In particular, notes that differ by the associated interval have primary fingerings that often differ on only one key. The example in Figure 1 is such an instrument. It has the property that keys $4, 5, 6$ often allow low cost movement of two-note intervals and key 7 often allows low cost movement of one-note intervals. For example, the following pairs of notes all differ by one key among the keys $4, 5, 6$: C_1 and D_1; $C\#_1$ and $D\#_1$; and D_1 and E_1. And the fingerings for C_1 and $C\#_1$; and D_1 and $D\#_1$ differ only on key 7. Although this is not evident from just the first two octaves of the instrument in Figure 1, the primary fingerings on keys $1, 2, 3$ have been similarly chosen so that movements of one octave always require the movement of only one finger. (It turns out that this instrument is optimal using primary fingerings for playing most standard scales consisting of intervals of size 1 and 2.) The trick behind the construction of basic instruments is the use of Gray codes, which were originally used in error correction of digital communications (see [10] and [23]).

Let us briefly describe our main results. We begin by enumerating all the basic instruments and calculating the costs of each (using primary and free fingerings) on several collections of note sequences, as discussed above. This calculation is carried out with a piece of software written by the author. We show that one of these basic instruments (instrument 7.1) is an optimal partitionable instrument, using primary fingerings, on a variety of scales: major, melodic minor (ascending and descending), whole tone, and diminished, for example. We find another basic instrument is an optimal partionable instrument, using primary fingerings, over the chromatic scales. And for all collections of fixed intervals (except those of size 5 and 7), we find basic instruments that are optimal partitionable instruments, using primary fingerings. (We also present optimal partitionable, but non-basic, instruments for collections of fixed intervals of size 5 and 7.) From these fixed interval results, we find optimal partitionable instruments for diminished 7 arpeggios and augmented triad arpeggios. In addition, we present a few cases of basic instruments that are optimal partitionable instruments using free fingerings. Thus the basic instruments appear to be a good source of low cost (easy-to-play) instruments. We also report the optimal basic instruments for the harmonic minor and pentatonic scales, and various other types of arpeggios. Furthermore, we find that instrument 7.1 is the optimal basic instrument, using primary fingerings, over both collections of melodies. We show that all the optimal partitionable instruments we present cannot be improved upon by considering partitionable instruments with more than seven keys. Hence our decision to consider instruments with seven keys does not appear to be an important limitation for a range of eight octaves. We also compare our designs with some existing instuments (e.g., the clarinet and saxophone) and find that, in almost all cases, our new instruments have significantly lower cost on all collections of note sequences. We also show the robustness of our results for instruments 1.2 and 7.1 by considering more general cost functions where the cost is only required to increase with the number of finger movements.

In short, the main contributions of this paper are the discovery of an important class of special instruments (the basic instruments), some new suggestions for good fingering systems, and, for future research, a general methodology and some benchmarks to use in searching for better fingering systems.

Remark 1 ***A practical implementation.*** *The Akai EWI and Yamaha WX, mentioned above, have been sold for a number of years and are used by professional musicians. (The author plays the EWI as a hobbiest.) They allow the user to select from several fingering systems, all of which are partitionable versions of fingering systems on trADititional instruments such as the saxophone or trumpet. However, instead of having three keys* $1, 2, 3$ *that select the octave as in our instruments, both have a row of keys (actually rollers on the EWI) along which the musician slides a thumb to select the octave. Hence there is one key for each octave in the instrument's range. The cost of moving from one octave to an adjacent one on these keys can reasonably be measured as a single finger movement, comparable in cost to pressing or releasing a single key. (The movement is from side to side by a thumb instead of an up and down movement of a finger.) Hence the approach to measuring costs that we use for our seven-key instruments exactly carries over to the way the EWI and WX are designed. The fingering systems introduced in this paper could easily be implemented in this way resulting in instruments with only four keys to be pressed for the pitch classes in combination with the row of octave keys. (See Remark 6.) (In fact, the EWI and WX could be reprogrammed to implement our systems.) Hence a musician would need to learn only* 12 *primary fingerings for the pitch classes in order to play the entire eight octave range (since the octave selection is greatly simplified). Furthermore, the four fingers for the pitch-class keys could be the index and middle fingers of both hands, which would eliminate the ergonomic difficulties inherent in using weaker fingers with less independence of movement. (E.g., see [14] and [18].) These tricks should result in instruments that are even easier to play. We present our results in this paper using the seven-key instrument model because the definitions and analysis are a bit nicer.*

We next discuss some related work. To begin, the results in this paper extend the results in [13]. The general definition of an instrument design problem and the methodology for comparing instruments by computing costs over common sequences of notes were introduced in [13]. The specific problem of finding an optimal fingering system for collections of scales consisting of adjacent intervals of size 1 and 2 was considered in [13]. Optimal instruments were presented for both partitionable and non-partitionable instruments using only primary fingerings. They are closely related to instruments 1.2 and 7.1 in this paper. The current paper adds computer search to the purely mathematical approach for finding optimal instruments in [13], which allows us to consider more general sequences of notes (other scales, the arpeggios, and intervals) and the costs of allowing alternate fingerings. This paper also adds consideration of sequences from samples of Western music melodies. The general class of basic instruments and the analysis of that class are also new to this paper. Furthermore, this paper addresses the issue of alternate cost functions.

The Bleauregard [1], mentioned above, uses a fingering system that, like our designs, has one finger per key, is partitionable, and is based on Gray codes. Its system is closely related to one of the designs presented in this paper (instrument 1.2); both are optimal for collections of chromatic scales. An interesting psychological justification for the Bleauregard system, called the helical model (see [8] and [27]), is discussed in [1].

Sayegh [24] studied the problem of finding efficient fingerings for playing sequences of notes on the guitar, where issues of alternate fingerings, selecting the finger to use, and the quality of the sound were considered. Sayegh considered using expert systems, optimization, and a connectionist approach. When one is given a single sequence of notes and a specific instrument (including many traditional instruments) with alternate fingerings for some notes, Worrall and Sharp [30] obtained a patent on a process for finding optimal fingerings. When one is given a sequence of notes to be played on the piano, Parncutt et al [20], Hart, Bosch, and Tsai [12], and Kasimi, Nichols, and Raphael [15] explored a dynamic programming approach that searches for the optimal finger to use for each note. (We use a similar approach when evaluating the cost of using alternate fingerings in this paper.)

A related problem of designing easy-to-play keyboard instruments has also been widely studied. A survey of many such designs can be found in Keislar [16]. Examples include the work of Bosanquet, Henfling, Janko, Fokker, Wilson, and Wesley. In this work, ease-of-play is achieved through designs where any given sequence of notes (or chord) is played with the same finger movements under any transposition. (This is accomplished by designs with many more keys than a piano with the same range.) This property does not hold for the designs presented in this paper where ease-of-play refers to total finger movements for given sequences of notes played over all transpositions (and our instruments have many fewer keys than comparable keyboard instruments).

In the non-musical realm of typing, the Dvorak keyboard [9] was developed to be more efficient, in terms of finger movements, than the standard QWERTY keyboard layout. The typing keyboard design problem has also been studied using optimization in [3] and [21]. The modern stenotype machine is a yet more efficient device for typing. Interestingly, it requires the operator to typically press more than one key at a time, like our instruments in this paper; with this device, experts can type over 200 words per minute, far faster than with a traditional one-key-per-letter layout.

The paper is organized as follows. In Section 2.1 we present a precise definition of the instruments we consider. In Section 2.2 we define the cost measure we use to compare different instruments. In Section 2.3 we define the collections of note sequences we use in our analysis. In Section 2.4 we define the basic instruments. Section 3 and the appendix contain our main results. In Section 4 we show the robustness of one of our best instruments by considering more general cost functions. We also discuss how a more precise cost function could be discovered by outlining a possible experiment. Section 5 summarizes our results, and some open questions are presented in Section 6. Finally, an appendix (found on-line) contains proofs of the mathematical statements in the paper and some additional results addressing when some of our instruments are optimal. Also on-line is the Excel spreadsheet that contains the VBA code used for the computer calculations in the paper.

2 Definitions

In this section we define the general instruments we consider. We discuss how we compare two different instruments. And we introduce a special class of instruments on which our later analysis focuses.

2.1 The general instruments

Let us describe the instruments we study. Our instruments play sequences of notes, one at a time, from the Western chromatic scale. A note is written $N(i)$, where i is an integer and $N(i+1)$ is a half step higher than $N(i)$. We sometimes refer to a note simply by its index i. For convenience, we assume our instruments can play all the notes $N(0), \ldots, N(95)$. This is called the instrument's *range*. An instrument might also be able to play some notes outside this range (e.g., $N(-1)$ or $N(96)$).

The actual pitch to which $N(0)$ corresponds can be set arbitrarily. The range size we have chosen turns out to be convenient given the definitions to follow and is reasonable in practice as it is slightly larger than a piano's range. The approach taken in this paper can easily be adapted to ranges of different size.

We let *octave* 1 denote the 12 notes $N(0), \ldots, N(11)$; we let *octave* 2 denote the 12 notes $N(12), \ldots, N(23)$; and so on. We let *pitch class* 1 denote the notes $N(0), N(12), \ldots, N(84)$; we let *pitch class* 2 denote the notes $N(1), N(13), \ldots, N(85)$; and so on. Notes in the same pitch class are separated by an integral number of octaves and have a similar sound quality. Hence our instruments have eight octaves and 12 pitch classes.

An instrument has a set of seven keys, labeled $1, \ldots, 7$. Any subset of the keys is called a *fingering*, hence an instrument has $2^7 = 128$ different fingerings.

An instrument has a *mapping* from the fingerings to the notes such that every note in the range has at least one fingering mapped to it. Observe that seven is the minimum possible number of keys so that an instrument can have at least one fingering mapped to each note in the range. (As we show in Section 3, this restriction to seven keys is often not a limitation.) This mapping is described by an *instrument matrix*, which has seven columns and 128 rows; and all its entries are 0s and 1s. The columns of the matrix are labeled by the keys. Each row of the matrix is labeled by a note such that every note in the range occurs at least once, and two or more rows can have the same label. (We allow notes outside the range to also be labels.) Every row of the matrix describes a different fingering, where the 1s indicate, by the corresponding column labels, the keys in the fingering. Each fingering is mapped to the corresponding row's label. (Hence, a musician plays a note by selecting a row labeled by that note and pressing the keys indicated by the 1s in the row.) The order of the rows in such a matrix is not important, but we assume, for convenience, that the rows are in ascending order by the note labels. (Observe that this ordering is not necessarily unique, since we allow more than one fingering to be mapped to the same note.)

Since each note $N(k)$ in the range can have more than one fingering mapped to it, we designate for each $N(k)$ a special fingering, called the *primary fingering*, that is mapped to $N(k)$. Its row in the instrument matrix is denoted $pr(N(k))$. These are the fingerings that a musician would first learn and would typically use when playing. Non-primary fingerings for $N(k)$ (if any) are called its *alternate fingerings*. These fingerings can be used on occassion by a musician to make certain passages easier to play. Primary and alternate fingerings play an important role, for example, for woodwinds. Observe that for notes playable by an instrument, but outside its range, we do not specify primary or alternate fingerings. (The reason for this is due to the specific primary fingerings that we define later in the paper.)

Examples of the instrument matrices (i.e., the first two octaves) for two instruments are given in Figure 2. The matrix for instrument 7.1 is given on the left of the figure. The matrix for instrument 9 is given to the right of the vertical double lines (The three matrices to the right of each instrument matrix describe the underlying structure of the matrix and will be discussed in Section 2.4.) Only the rows for the first two octaves of the matrices are shown. The rows with large font are primary fingerings, those with small font are alternate fingerings.

We next define a special type of instrument that will be a main focus of this paper. (As discussed in the introduction, these special instruments should prove easy to play.) An instrument is called *partitionable* if its primary fingerings satisfy the following:

- Two notes have the same primary fingerings on keys $1, 2, 3$ if and only if they are in the same octave.
- Two notes have the same primary fingerings on keys $4, 5, 6, 7$ if and only if they are in the same pitch class.

In summary, an instrument has the following properties: an eight octave range; seven keys; a mapping from the fingerings to the notes such that every note in the range has at least one fingering mapped to it; and a choice of a primary fingering for each note in the range. The mapping is described by an instrument matrix with row and column labels. Due to our ease-of-play objective, we focus our attention on partitionable instruments.

Referring to Figure 2, we show (in Section 3) that instrument 7.1 is optimum for primary fingerings over all partitionable instruments for playing a variety of scales that include major scales, melodic minor scales, whole tone scales, and diminished scales (see [25] for definitions), and is also optimum over both of our collections of melodies. We show instrument 9 is optimum for primary fingerings over an important subclass of partitionable instruments (called *basic*) for playing major and minor arpeggios.

2.2 Comparing instruments

In this section we consider how to compare a set of different instruments so we can find the "best" in the set. The basic idea is to specify a collection of note sequences and to see for each instrument how many finger movements are required to play all the sequences. Instruments with the minumum number of movements are considered the best. Because we allow alternate fingerings for some notes, some care is required in making this notion precise.

We begin by introducing a notion of "distance between fingerings," which is a measure of the difficulty of playing one fingering followed by another.

Consider an instrument with instrument matrix M. Let f_1 and f_2 be two rows of M, corresponding to two fingerings. We define the *Hamming distance from f_1 to f_2*, denoted $Hdist(f_1, f_2)$, as follows:

$$Hdist(f_1, f_2) = \sum_{i=1}^{7} |f_1(i) - f_2(i)| \,.$$

In short, the Hamming distance between two fingerings is the number of keys that change from pressed to not pressed, or vice versa, as a musician plays one fingering followed by the other. (The Hamming distance was first defined in [11] in the context of information theory.) For an example, consider the matrix for instrument 7.1 in Figure 2. Observe that the distance between the fingerings for $N(0)$ and $N(1)$ is 1 and the distance between the fingerings for $N(1)$ and $N(2)$ is 2. (Given the instruments and sequences we consider, the range of the $Hdist$ function is $\{1, 2, 3, 4, 5\}$ in this paper.)

Let us remark that the $Hdist$ function is only one possible measure of the difficulty of playing two notes, one after the other. It captures the notion that the difficulty of playing two consecutive notes increases as more fingers must change positions, especially under the assumption of Remark 1, where only five strong and independent fingers are needed. However, it may not be the case, for example, that moving two fingers is twice as difficult as moving one finger, or that moving three fingers is three times as difficult as moving one finger, as the

Instrument 7.1

	1234567
$N(0)$	0001000
$N(1)$	0001001
$N(2)$	0001100
$N(3)$	0001101
$N(4)$	0000100
$N(5)$	0000101
$N(6)$	0000110
$N(7)$	0000111
$N(8)$	0001110
$N(9)$	0001111
$N(10)$	0001010
N(10)	1000000
$N(11)$	0001011
N(11)	1000001
$N(12)$	1001000
N(12)	0000010
$N(13)$	1001001
N(13)	0000011
$N(14)$	1001100
$N(15)$	1001101
$N(16)$	1000100
$N(17)$	1000101
$N(18)$	1000110
$N(19)$	1000111
$N(20)$	1001110
$N(21)$	1001111
$N(22)$	1001010
N(22)	1100000
$N(23)$	1001011
N(23)	1100001

$\mathbf{B(12, 3, 0)}$

	123
0	000
12	100
24	110
36	010
48	011
60	111
72	101
84	001

$\mathbf{B(2, 3, -2)}$

	456
−2	000
0	100
2	110
4	010
6	011
8	111
10	101
12	001

$\mathbf{B(1, 1, 0)}$

	7
0	0
1	1

Instrument 9

	1234567
$N(0)$	0000000
$N(1)$	0000010
$N(2)$	0000011
$N(3)$	0000001
$N(4)$	0001000
$N(5)$	0001010
$N(6)$	0001011
$N(7)$	0001001
$N(8)$	0001100
$N(9)$	0001110
$N(10)$	0001111
$N(11)$	0001101
$N(12)$	1000000
N(12)	0000100
$N(13)$	1000010
N(13)	0000110
$N(14)$	1000011
N(14)	0000111
$N(15)$	1000001
N(15)	0000101
$N(16)$	1001000
$N(17)$	1001010
$N(18)$	1001011
$N(19)$	1001001
$N(20)$	1001100
$N(21)$	1001110
$N(22)$	1001111
$N(23)$	1001101

$\mathbf{B(12, 3, 0)}$

	123
0	000
12	100
24	110
36	010
48	011
60	111
72	101
84	001

$\mathbf{B(4, 2, 0)}$

	45
0	00
4	10
8	11
12	01

$\mathbf{B(1, 2, 0)}$

	67
0	00
1	10
2	11
3	01

Figure 2: Two instrument designs.

$Hdist$ function implies. We discuss some more general measures of difficulty in Section 4, where we also suggest how the notion of difficulty could be quantified and studied experimentally.

Let $\{s_1, \ldots, s_q\}$ be a sequence of notes in the range of the instrument (where $q \geq 2$). We define the *cost of playing* $\{s_1, \ldots, s_q\}$ *with primary fingerings* to be

$$Cost_{pr}(s_1, \ldots, s_q) = \sum_{i=1}^{q-1} Hdist(pr(s_i), pr(s_{i+1})).$$

Finally, for any given collection C of sequences of notes in the range of the instrument, we define:

$$TotalCost_{pr}(C) = \sum_{S \in C} Cost_{pr}(S).$$

This is a measure of how difficult it is to play all the sequences in C using only the primary fingerings.

We let fr denote an arbitrary function from each note in the range of an instrument to a single row of M labeled by that note. Hence the fr function selects a fingering for each note; the pr function is an example of an fr function where the primary fingering is always selected. We now define the *cost of playing* $\{s_1, \ldots, s_q\}$ *with free fingerings* to be

$$Cost_{fr}(s_1, \ldots, s_q) = \min_{fr} \sum_{i=1}^{q-1} Hdist(fr(s_i), fr(s_{i+1})),$$

where the min is taken over all fr functions for the instrument. In words, the cost of playing a sequence of notes with free fingerings is the minimum number of finger movements required to play the sequence. Observe that determining this cost for a given sequence of notes is an optimization problem.

As above, for any given collection C of sequences of notes in the range of the instrument, we define:

$$TotalCost_{fr}(C) = \sum_{S \in C} Cost_{fr}(S).$$

This is a measure of how difficult it is to play all the sequences in C where all fingerings are allowed.

We now define the problem we want to solve. For a collection C of sequences of notes and a function F, we are interested in the following two problems:

- The *instrument design problem with primary fingerings* is to find an instrument that minimizes $TotalCost_{pr}(C)$;
- The *instrument design problem with free fingerings* is to find an instrument that minimizes $TotalCost_{fr}(C)$.

In this paper we solve these problems for restricted sets of instruments and restricted collections of sequences of notes. These restrictions are the topics of the following two subsections. How to efficiently solve these problems in their full generality (over all instruments and arbitrary collection of sequences) is an open problem. (As noted in Section 1, enumeration appears to be impractical, in general, due to the immense number of different instruments.)

2.3 The note sequences

In this section we define the collections C of note sequences that we consider in this paper. As discussed in the introduction, because common building blocks of Western music are scales and arpeggios, we have chosen to include some commonly occurring examples in our collections C. We have also included the simple collections containing intervals of a given size. From these nicely structured sequences we find evidence, when we analyze our results in Section 3 and in the appendix, of the importance of the special class of "basic" instruments on which we focus our attention. Finally, we consider note sequences from the melodies in two samples of Western music. These samples come from work of Vos and Troost [28] and consist mostly of melodies by 13 well-known European composers and folk music from several traditions. In fact, the results in [28] were reported as frequency distributions for the intervals between adjacent notes in the sequences (which work well for our purposes when combined with our results for fixed interval sequences). We report these distributions below. We begin by listing the names of the collections of structured sequences we consider in Figure 3.

The forms of the sequences of notes in each collection are described in Figure 4. For example, the first sequence in the MajSc collection contains the notes with indices given in the figure. That is, the first sequence in MajSc is

$$N(0), N(2), N(4), N(5), N(7), N(9), N(11), N(12),$$

which is the major scale starting at $N(0)$. The second sequence in MajSc is obtained by adding 1 to each index in the first sequence. That is, the second sequence is

$$N(1), N(3), N(5), N(6), N(8), N(10), N(12), N(13),$$

which is the major scale starting at $N(1)$. The MajSc collection contains all such scales with starting notes in the first seven octaves in the instrument's range. (Hence the highest starting note is $N(83)$.)

In general, each collection of sequences contains all the sequences starting at the notes in the first seven octaves. Hence, each collection contains $12 \cdot 7 = 84$ sequences. The highest note in each collection of scales, each collection of arpeggios, and the collection of intervals of size 12 is the highest note in the instrument's range: $N(95)$. The other collections of intervals do not contain $N(95)$.

Each pair of notes in the collection Int$i$ is called an interval of *size* i.

Abbreviations for collections of note sequences	
MajSc:	Major scales
MinScH:	Harmonic minor scales
Pen:	Pentatonic scales
Chr:	Chromatic scales
MajArp:	Major arpeggios
MinArp:	Minor arpeggios
Dom7:	Dominant 7th arpeggios
Int1:	Intervals of size 1
Int2:	Intervals of size 2
⋮	⋮
Int12:	Intervals of size 12
Eur:	Melodies by European composers
Folk:	Melodies in folk songs

Figure 3: Abbreviations for collections of note sequences

First sequences (by note indices)													
MajSc	0	2	4	5	7	9	11	12					
MinScH	0	2	3	5	7	8	11	12					
Pen	0	2	4	7	9	12							
Chr	0	1	2	3	4	5	6	7	8	9	10	11	12
MajArp	0	4	7	12									
MinArp	0	3	7	12									
Dom7	0	4	7	10	12								
Int1:	0	1											
Int2:	0	2											
⋮	⋮	⋮											
Int11:	0	11											

Figure 4: First sequences (by note indices)

Frequency distributions for Western music melodies													
Interval size	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Composers (%)	12.5	21	31.5	11	6.5	8.5	1	3	1	2	1	0	1
Folk songs (%)	18	17	39	12.5	7	4	0	1	.5	.5	0	0	.5

Figure 5: Frequency distributions for Western music melodies

The table in Figure 5 contains summary data from two samples of Western music melodies from [28]. For example, it states that 39% of all intervals between adjacent notes in the sample of folk song melodies have size 2. An interval size of 0 indicates two adjacent notes that are the same. Such occurences have zero cost in our model, since no finger movements are required. (We point out that the distribution for bass lines may be quite different, with larger intervals being more common.)

Observe that our collections range from sequences with many small intervals between adjacent notes (e.g., ChrSc, Int1, and the melodies) to sequences with larger such intervals (e.g., MajArp, MinArp, and Int11).

2.4 Instruments based on intervals

In this section we describe a special class of instruments, called *interval-based instruments*, along with an algorthm for choosing primary fingerings, which will yield a number of our optimal instruments. The idea is to construct, for a given choice of intervals, instruments that are likely to have low cost when playing sequences of notes containing many of those intervals between adjacent notes. At the end of the section we define the class of basic instruments.

Let us begin with an informal example of how we structure a mapping from the fingerings to the notes. Consider the instrument matrix for instrument 7.1 on the left side of Figure 2. It is constructed from three *interval matrices*, which are shown to its right in the figure and labeled $B(12,3,0)$, $B(2,3,-2)$, and $B(1,1,0)$. Notice that the interval matrices have column labels that correspond to column labels of the instrument matrix. In particular, $B(12,3,0)$ has column labels $1,2,3$; $B(2,3,-2)$ has column labels $4,5,6$; and $B(1,1,0)$ has column label 7. To obtain a fingering for a note, say $N(i)$, we select one row from each interval matrix such that the selected rows' labels sum to i. Then we construct the fingering by stringing together the three corresponding rows of 0s and 1s from the matrices. For example, to get a fingering for note $N(23)$, we can select the row of $B(12,3,0)$ labeled 12, the row of $B(2,3,-2)$ labeled 10, and the row of $B(1,1,0)$ labeled 1, since $12+10+1=23$. Stringing together the corresponding rows yields the row of instrument 7.1 labeled $N(23)$ (in the large font). Observe that we could have selected the row of $B(12,3,0)$ labeled 24, the row of $B(2,3,-2)$ labeled -2, and the row of $B(1,1,0)$ labeled 1; this yields the row of instrument 7.1 labeled $N(23)$ (in the small font), which is an alternate fingering for $N(23)$. (Below we describe the rule we use to select the primary fingerings.)

Let us examine the underlying logic of this system. Consider again instrument 7.1 in Figure 2. Observe first that each interval matrix has the property that the distance between adjacent rows is 1 (that is, the 0s and 1s differ in only one column). Observe next that the row labels for the interval matrix $B(12, 3, 0)$ are multiples of 12, the row labels for $B(2, 3, 0)$ are multiples of 2, and the row labels for $B(1, 1, 0)$ are multiples of 1, always in ascending order. (Hence the first number in an interval matrix's description is this multiplier. The second number is the matrix's number of columns and the third number is its smallest row label.) Suppose we are playing the note $N(6)$. Then the cost of next playing the note up 12 indices (i.e., an octave) is just 1, as are the costs of playing the notes up 1 or 2 indices. Although this is not the case for all starting notes, it is often the case, or close to being the case; hence instrument 7.1 has the nice property that playing, for example, the modes of major scales (consisting of intervals of size 1 and 2) has low total cost. Similarly, instrument 9 (also in Figure 2) has the property that playing consecutive notes that are 12, 4, or 1 step apart has low cost. This instrument has the property that playing arpeggios (consisting of intervals of size 3, 4, and 5) has low total cost. (We mean here that the scales and arpeggios are played as described in Section 2.3.)

We next present a formal description of our instruments. An *interval matrix* is one of the 0-1 matrices in Figure 6, with row labels as shown in the figure and arbitrary distinct column labels. (The displayed column labels from $1, 2, 3, 4$ will change in applications.) The row labels are determined by setting v to be an integer, called the *initial value*, and setting s to be a positive integer, called the *interval size*. Such a labeled matrix is denoted $B(s, t, v)$, where t is the number of columns. If $v = 0$, then we sometimes denote the matrix $B(s, t)$. See Figure 2 for specific examples of interval matrices. Observe that interval matrices satisfy the following properties:

- The entries are 0 or 1.
- For adjacent rows, say r_1 and r_2, we have $Hdist(r_1, r_2) = 1$. (This also holds for the first and last rows.)
- No two rows are identical.

Matrices satisfying these properties are called *Gray codes*. They play an important role in communications theory (see [10] and [23]). Gray codes with a given number of rows and columns need not be unique. We use a standard type of Gray code called *binary-reflected*. (For example, see [23] for the construction method.)

The special type of instruments we consider in this paper are constructed from a set of interval matrices as follows. Let

$$B = \{B_1 = B(s_1, t_1, v_1), \ldots, B_m = B(s_m, t_m, v_m)\}$$

$\mathbf{B(s,1,v)}$

$$\begin{array}{c|c|} & 1 \\ \cline{2-2} v & 0 \\ v+s & 1 \\ \cline{2-2} \end{array}$$

$\mathbf{B(s,2,v)}$

$$\begin{array}{c|c|} & 1\,2 \\ \cline{2-2} v & 0\,0 \\ v+s & 1\,0 \\ v+2s & 1\,1 \\ v+3s & 0\,1 \\ \cline{2-2} \end{array}$$

$\mathbf{B(s,3,v)}$

$$\begin{array}{c|c|} & 1\,2\,3 \\ \cline{2-2} v & 0\,0\,0 \\ v+s & 1\,0\,0 \\ v+2s & 1\,1\,0 \\ v+3s & 0\,1\,0 \\ v+4s & 0\,1\,1 \\ v+5s & 1\,1\,1 \\ v+6s & 1\,0\,1 \\ v+7s & 0\,0\,1 \\ \cline{2-2} \end{array}$$

$\mathbf{B(s,4,v)}$

$$\begin{array}{c|c|} & 1\,2\,3\,4 \\ \cline{2-2} v & 0\,0\,0\,0 \\ v+s & 1\,0\,0\,0 \\ v+2s & 1\,1\,0\,0 \\ v+3s & 0\,1\,0\,0 \\ v+4s & 0\,1\,1\,0 \\ v+5s & 1\,1\,1\,0 \\ v+6s & 1\,0\,1\,0 \\ v+7s & 0\,0\,1\,0 \\ v+8s & 0\,0\,1\,1 \\ v+9s & 1\,0\,1\,1 \\ v+10s & 1\,1\,1\,1 \\ v+11s & 0\,1\,1\,1 \\ v+12s & 0\,1\,0\,1 \\ v+13s & 1\,1\,0\,1 \\ v+14s & 1\,0\,0\,1 \\ v+15s & 0\,0\,0\,1 \\ \cline{2-2} \end{array}$$

Figure 6: The interval matrices.

be a set of m interval matrices such that $t_1 + \cdots + t_m = 7$ and the seven column labels are distinct. From these interval matrices we construct a single matrix, denoted $I(B)$, as follows. For every selection of one row r_i from each matrix B_i, add the row $r = (r_1, \ldots, r_m)$ to $I(B)$; and label it with the note $N(k)$, where k is equal to the sum of the row labels of $r_1, \ldots, r_m$ in $B_1, \ldots, B_m$, respectively. The seven columns of $I(B)$ inherit their labels, called $K(B)$, from the column labels in B. Observe that if we interpret the set $K(B)$ as keys, then $I(B)$ has 2^7 rows representing each fingering on $K(B)$. If $I(B)$ defines an instrument matrix (i.e., its row labels include $N(0), \ldots, N(95)$), then we say the instrument matrix is *interval-based*.

We next define an algorithm for determining the primary fingerings for an interval-based instrument. One motivating idea is that, for simplicity, primary fingerings should arise from rows of the interval matrices with nonnegative labels. Toward this end, we make the following definition.

Let $B = \{B_1 = B(s_1, t_1, v_1), \ldots, B_m = B(s_m, t_m, v_m)\}$, where the v_i are *not necessarily* equal to 0. Assume $I(B)$ defines an interval-based instrument. Consider a row $r = (r_1, \ldots, r_m)$ of $I(B)$. Then the fingering corresponding to r is called *nonnegative* if the row labels for $r_1, \ldots, r_m$ from $B_1, \ldots, B_m$ are all nonnegative. If every row label in $N(0), \ldots, N(95)$ has at least one nonnegative fingering mapped to it by $I(B)$, then we say $I(B)$ is *nonnegative*. Observe that the instrument matrices defined in Figure 2 are nonnegative.

Remark 2 *Let $I(B)$ be a nonnegative interval-based instrument matrix and let $B(s_i, t_i, v_i)$ be an arbitrary interval matrix in B. Then $B(s_i, t_i, v_i)$ must have a row with label 0 (since $N(0)$ is in the range), hence $v_i \leq 0$.*

We choose primary fingerings using the following algorithm.

Algorithm P Let $I(B)$ be a nonnegative interval-based instrument matrix. Put the interval matrices in B into decreasing order by their interval sizes (breaking ties with decreasing order by number of nonnegative row labels in each matrix). The primary fingering for a note $N(k)$ in the range is obtained by considering the matrices in this order and selecting, from each matrix in turn, the row with largest nonnegative label such that the sum of the labels of the latest and previously selected rows is at most k.

Example 3 *Consider instrument 7.1 in Figure 2. Let us implement Algorithm P to find the primary fingering for $N(11)$. First we put the interval matrices into the order $B(12,3,0)$, $B(2,3,-2)$, $B(1,1,0)$. From matrix $B(12,3,0)$ we select row label 0 because the other row labels are greater than 11. From matrix $B(2,3,0)$ we select row label 10, because it is the largest label such that $0 + 10 \leq 11$. Finally, from matrix $B(1,1,0)$ we select row label 1, because it is the largest label such $0 + 10 + 1 \leq 11$. The rows of the interval matrices corresponding to these three labels now determine the primary fingering for $N(11)$.*

The rationale for putting the interval matrices into decreasing order by their interval sizes is to, as much as possible, choose rows with small row labels, since rows near each other will tend to have small distances between them. But the best justification for the above algorithm is that it succeeds in producing partitionable instruments that are often optimum (as we see in Section 3 and in the appendix).

The following proposition states that Algorithm P always works.

Proposition 4 *If $I(B)$ is a nonnegative interval-based instrument matrix, then Algorithm P defines a selection of primary fingerings for $I(B)$.*

Proof: See Section 7 in the appendix. ■

Finally, we define the class of instruments that will be a main focus in the rest of the paper.

An instrument is called *basic*, if it satisfies the following two properties:

P1 The instrument matrix is interval-based and nonnegative, and $B(12, 3)$ is one of the interval matrices.

P2 The primary fingerings are given by Algorithm P.

Remark 5 *All basic instruments are partitionable (when the three keys associated with $B(12, 3)$ are $1, 2, 3$). However, there are partitionable instruments that are not basic; two examples, called OptInt5 and OptInt7, are given below.*

Remark 6 *Let us point out how the basic instruments can be converted to an instrument as described in Remark 1, where the thumb is used to select the octave in place of keys 1,2,3. Simply label the rows of $B(12, 3)$ with, say, a, b, c, d, e, f, g, h, and replace the keys $1, 2, 3$ on the instrument with keys a, b, c, d, e, f, g, h. Then, for any note on any instrument matrix, the converted fingering is obtained from the row of the matrix by using the key from a, b, c, d, e, f, g, h that labels the specified combination of the keys $1, 2, 3$ in $B(12, 3)$.*

We enumerate the basic instruments in the next section and show they have a number of nice properties.

3 Main results

In this section we present our main results. We begin by enumerating all the basic instruments in Proposition 7. We then compute the total costs for each basic instrument over each collection's note sequences. We show that the basic instruments contain optimal partitionable instruments in a number of situations. We also show that all the optimal partitionable instruments discussed in this section have the property that no partitionable instrument with more than seven keys can have lower total cost.

Proposition 7 *Figure 7 contains all the basic instruments. (Since each instrument uses the interval matrix $B(12, 3)$, this matrix is not shown in the figure.)*

Names	Interval matrices		
1	$B(1,4)$		
2	$B(1,3)$	$B(4,1)$	
3	$B(1,3)$	$B(5,1)$	
4	$B(1,3)$	$B(6,1)$	
5	$B(1,3)$	$B(7,1)$	
6	$B(1,3)$	$B(8,1)$	
7	$B(2,3)$	$B(1,1)$	
8	$B(1,2)$	$B(3,2)$	
9	$B(1,2)$	$B(4,2)$	
10	$B(1,2)$	$B(4,1)$	$B(4,1)$
11	$B(1,2)$	$B(4,1)$	$B(5,1)$
12	$B(1,2)$	$B(4,1)$	$B(6,1)$
13	$B(1,2)$	$B(4,1)$	$B(7,1)$
14	$B(1,2)$	$B(4,1)$	$B(8,1)$

Names	Interval matrices			
15	$B(2,2)$	$B(1,1)$	$B(4,1)$	
16	$B(2,2)$	$B(1,1)$	$B(5,1)$	
17	$B(2,2)$	$B(1,1)$	$B(6,1)$	
18	$B(2,2)$	$B(1,1)$	$B(7,1)$	
19	$B(2,2)$	$B(1,1)$	$B(8,1)$	
20	$B(3,2)$	$B(1,1)$	$B(1,1)$	
21	$B(3,2)$	$B(1,1)$	$B(2,1)$	
22	$B(4,2)$	$B(1,1)$	$B(2,1)$	
23	$B(1,1)$	$B(1,1)$	$B(3,1)$	$B(6,1)$
24	$B(1,1)$	$B(2,1)$	$B(2,1)$	$B(6,1)$
25	$B(1,1)$	$B(2,1)$	$B(3,1)$	$B(5,1)$
26	$B(1,1)$	$B(2,1)$	$B(3,1)$	$B(6,1)$
27	$B(1,1)$	$B(2,1)$	$B(3,1)$	$B(7,1)$
28	$B(1,1)$	$B(2,1)$	$B(4,1)$	$B(4,1)$
29	$B(1,1)$	$B(2,1)$	$B(4,1)$	$B(5,1)$
30	$B(1,1)$	$B(2,1)$	$B(4,1)$	$B(6,1)$
31	$B(1,1)$	$B(2,1)$	$B(4,1)$	$B(7,1)$
32	$B(1,1)$	$B(2,1)$	$B(4,1)$	$B(8,1)$

Names	Interval matrices		
1.1	$B(1,4,-1)$		
1.2	$B(1,4,-2)$		
1.3	$B(1,4,-3)$		
1.4	$B(1,4,-4)$		
3.1	$B(1,3,-1)$	$B(5,1)$	
4.1	$B(1,3,-1)$	$B(6,1)$	
4.2	$B(1,3,-2)$	$B(6,1)$	
5.1	$B(1,3,-1)$	$B(7,1)$	
7.1	$B(2,3,-2)$	$B(1,1)$	
7.2	$B(2,3,-4)$	$B(1,1)$	
8.1	$B(1,2,-1)$	$B(3,2)$	
9.1	$B(1,2)$	$B(4,2,-4)$	
17.1	$B(2,2,-2)$	$B(1,1)$	$B(6,1)$
22.1	$B(4,2,-4)$	$B(1,1)$	$B(2,1)$

Figure 7: The basic instruments.

Proof: See Section 7. ■

Observe that the instruments 1-32 in Figure 7 have the property that all the interval matrices have $v_i \geq 0$. For each remaining instrument, at least one interval matrix has $v_i < 0$; these instrument names are derived from the related instruments 1-32.

In order to compare the basic instruments with each other and some preexisting instruments, a computer program was written using VBA in Excel. The code takes as input the list of note sequences given in Figure 3 and the basic instruments given in Figure 7. The code outputs the total cost of playing each collection of sequences on each instrument, using both primary and free fingerings. The calculation of the total costs using free fingerings (which is an optimization problem) was accomplished using a dynamic programming approach (e.g., see [7]). The one exception to this are the total costs for Eur and Folk. These were computed by taking, for each basic instrument, a weighted average of the total costs for Int1 ... Int12 using the frequencies in Figure 5 for the weights. (There are no total costs for free fingerings because the actual sequences are not readily available.) The results are summarized in Figures 8, 9, and 10.

Figure 8 contains examples of optimal basic instruments for each collection of sequences using both primary and free fingerings. (Rather than report all optimal instruments in this table, only those that were optimal in the most situations have been listed.) Figure 9 contains the complete output of total costs for some of the best instruments in Figure 8. Figure 10 contains the same information for five preexisting instruments, so that comparisons can be made. The clarinet model we used has 24 keys and the saxophone model we used has 22 keys. Both models, with primary and alternate fingerings, were obtained from [4]. The EVI abbreviation refers to the Electronic Valve Instrument. This is a fingering system option for the EWI; it is based on trumpet fingerings and is essentially equivalent to basic instrument 25. For consistency, we used a seven key partitionable version of the Bleauregard system. The final instrument, labeled Optimal for Int5, is discussed below and defined in the appendix. The total costs for the clarinet were computed over four octaves of its range (from E_1 to $D\#_4$) and then scaled to eight octaves for comparison. The total costs for the saxophone were computed over three octaves of its range (from D_1 to $C\#_3$) and then scaled to eight octaves. The total costs for all the other instruments were computed over three octaves and then scaled to eight octaves.

Let us first point out some instruments that stand out from Figure 8. Clearly instrument 7.1 is noteworthy. It is an optimal basic instrument for MajSc, MajAcSc, MinHSc, MajHSc, OctSc, Int2, and Int10 using both primary and free fingerings. It is also optimal for PenSc using primary fingerings and nearly optimal for Pen using free fingerings (see Figure 9). Instrument 7.1 is also optimal for Eur and Folk using primary fingerings. Another important instrument is 1.2. It is an optimal basic instrument for ChrSc, Int1, and Int11 using both primary and free fingerings. Instrument 9 is an optimal basic instrument for MajArp, MinArp, Maj7Arp, Int4, and Int8. Instrument 8 is an optimal basic instrument for Dom7Arp, Min7Arp, HaDimArp, Int3, and Int9 using both primary and free

Optimal basic instruments				
	Primary fingerings		Free fingerings	
	Instruments	Total costs	Instruments	Total costs
MajSc:	7.1	756	7.1	728
MinScH:	7.1	924	7.1	896
Pen:	8; 21; 7.1	756	21	721
Chr:	1.2	1092	1.2	1064
MajArp:	9	560	9	504
MinArp:	9	560	9	504
Dom7:	8; 21	644	8	581
Int1:	1.2	91	1.2	84
Int2:	7.1	98	7.1	84
Int3:	8; 21	105	8; 21	98
Int4:	9; 14	140	9	84
Int5:	16; 18	175	16	140
Int6:	4	126	4	112
Int7:	16; 18	189	18	140
Int8:	9; 14	168	14	84
Int9:	8; 21	147	8; 21	140
Int10:	7.1	154	7.1	154
Int11:	1.2	161	1.2	161
Int12:	all	84	all	84
Eur:	7.1	144		
Folk:	7.1	122		

Figure 8: Optimal basic instruments

fingerings. (Instrument 21 is very similar to Instrument 8.)

Next, let us consider some intuition behind the performance of these important basic instruments. First, observe that each sequence in our collections contains a particular distribution of interval sizes between adjacent notes:

MajSc, MajAcSc: 5 intervals of size 2 and 2 intervals of size 1.

MinHSc, MajHSc: 3 intervals of size 2, 3 intervals of size 1, and 1 interval of size 3.

OctSc: 4 intervals of size 2 and 4 intervals of size 1.

PenSc: 3 intervals of size 2 and 2 intervals of size 3.

ChrSc: 12 intervals of size 1.

MajArp, MinArp: 1 interval of size 3, 1 interval of size 4, and 1 interval of size 5.

Maj7Arp: 2 intervals of size 4, 1 interval of size 3, and 1 interval of size 1.

Dom7Arp, Min7Arp, HaDimArp: 1 interval of size 2, 2 intervals of size 3, and 1 interval of size 4.

For these examples, we see that scales and arpeggios with the same interval distributions have the same optimal instrument and the same total cost for primary and free fingerings. The proofs of (upcoming) Proposition **??** and Remark 8 give a condition when this is true in general.

We observe, in general, that basic instruments whose interval matrices have interval sizes that are close to a commonly occurring interval size in a sequence of notes, tend to have low cost on that sequence. For example, observe that instrument 7.1 has $B(2,3)$ as an interval matrix and it has low cost on those sequences that contain many intervals of size 2: MajSc, MajAcSc, MinHSc, MajHSc, OctSc, PenSc, Int2, Eur, and Folk. It also has low cost on Int10 because the use of Gray codes for interval matrices makes playing intervals of size 10 also have low cost. Instrument 1.2 has $B(1,4,-2)$ as an interval matrix so it has low cost on ChrSc, Int1, and Int11. Instrument 9 has $B(4,2)$ as an interval matrix so it has low cost on MajArp, MinArp, and Maj7Arp, which contain intervals of size 4 (and close to size 4), and Int4 and Int8. Similarly, Instrument 8 has $B(3,2)$ as an interval matrix so it has low cost on Dom7Arp, which contains intervals of size 3 (and close to size 3), and Int3 and Int9.

Another interesting feature of the optimal basic instruments is that instruments 1.2 and 7.1 have better total costs in many situations than instruments 1 and 7. Consider instrument 7.1. One of its interval matrices is $B(2,3,-2)$. The result is that the primary fingerings on keys $4,5,6$ in each octave are constructed from the middle six rows of matrix $B(s,3,v)$ in Figure 6. Observe that these six rows are themselves a Gray code matrix, hence the cost of moving from one octave to the next on an interval of size 2 using primary fingerings is lower for instrument 7.1 than for instrument 7. For example, consider playing $N(10)$ then $N(12)$ on instrument 7.1 in Figure 2. The cost on keys 4,5,6 is 1 due to this six-row gray code in the middle of $B(2,3,-2)$. If we had used $B(2,3,0)$ instead (yielding instrument 7), this cost would have been 3. A similar argument holds for instruments 1.2 compared to instrument 1 on intervals of size 1. This provides a justification for using Algorithm P for selecting primary fingerings based on nonnegative instrument matrices.

Total costs for some of the best basic instruments								
	1.2		7.1		8		9	
	Prim.	Free	Prim.	Free	Prim.	Free	Prim.	Free
MajSc:	1092	1064	756	728	1204	1085	1428	1372
MinScH:	1008	980	924	896	1092	973	1260	1204
Pen:	924	896	756	728	756	749	1092	1036
Chr:	1092	1064	1596	1568	1764	1701	1428	1372
MajArp:	644	581	728	665	644	581	560	504
MinArp:	644	567	728	651	644	581	560	504
Dom7:	756	728	756	728	644	581	756	700
Int1:	91	84	133	126	147	84	119	105
Int2:	182	168	98	84	182	175	238	210
Int3:	189	168	231	210	105	98	189	147
Int4:	196	168	196	168	252	189	140	84
Int5:	259	190	301	232	287	280	231	189
Int6:	322	212	238	128	210	203	322	294
Int7:	273	204	315	246	301	294	245	231
Int8:	224	196	224	196	280	232	168	168
Int9:	231	217	273	259	147	140	231	231
Int10:	238	238	154	154	238	231	294	294
Int11:	161	161	203	203	217	169	189	189
Int12:	84	84	84	84	84	84	84	84
Eur:	153		144		161		170	
Folk:	140		122		144		161	

Figure 9: Total costs for some of the best basic instruments

We see in all cases that using free fingerings results in lower total cost than using primary fingerings.

Unfortunately, as Figure 9 shows, none of our best basic instruments completely dominates the others. In particular, instrument 7.1 is clearly the best on MajSc, MajAcSc, MinHSc, MajHSc, OctSc, PenSc, Eur, and Folk; instrument 1.2 is clearly the best on ChrSc; instrument 8 is clearly the best on Dom7Arp, Min7Arp, and HaDimArp; and instrument 9 is clearly the best on MajArp, MinArp, and Maj7Arp. Presumably, the final choice of an instrument design based on these computations would be influenced strongly by the type of sequences one expects to be playing most frequently.

Figures 9 and 10 demonstrate that the four best basic instruments essentially dominate the clarinet and saxophone on total cost (and, of course, have far fewer keys) on every collection of note sequences. Instruments 1.2, 7.1, and 8

Total costs for other instruments										
	Clarinet		Saxophone		EVI (Inst. 25)		Bleauregard		Opt for Int5	
	Prim.	Free	Prim.	Free	Prim.	Free	Prim.	Free	Prim.	Free
MajSc:	1430	1374	1477	1383	1232	1204	1092	1064	1372	1358
MinScH:	1402	1351	1463	1369	1232	1176	1064	1008	1428	1414
Pen:	1220	1155	1239	1183	952	882	1036	924	924	903
Chr:	1878	1869	2037	1887	1932	1904	1092	1064	2772	2758
MajArp:	931	873	966	865	616	560	728	599	504	483
MinArp:	917	861	945	858	616	567	728	602	504	476
Dom7:	1127	1066	1169	1064	812	756	924	788	756	742
Int1:	145	133	147	126	161	140	91	84	231	221
Int2:	201	184	182	175	182	154	182	168	182	168
Int3:	257	238	242	228	203	175	245	189	189	175
Int4:	310	287	287	259	224	196	252	200	196	182
Int5:	350	327	329	294	189	168	231	200	119	91
Int6:	380	364	350	319	266	238	238	217	322	270
Int7:	406	392	385	347	203	182	245	203	133	133
Int8:	436	420	368	326	252	224	280	217	224	196
Int9:	441	422	354	312	245	217	287	221	231	203
Int10:	446	432	336	301	238	210	238	210	238	210
Int11:	450	429	329	287	231	210	161	147	301	263
Int12:	441	425	252	214	84	84	84	84	84	84
Eur:	210		193		163		161		167	
Folk:	181		168		152		150		156	

Figure 10: Total costs for other instruments

dominate the EVI fingering system on all scales and arpeggios, except MajArp and MinArp. Instrument 9 dominates the EVI on MajArp and MinArp. Instrument 1.2 equals or slightly dominates the Bleauregard on total costs on scales and clearly dominates on arpeggios. The instrument Opt for Int5 (discussed in the appendix) is clearly the best instrument in the figures on MajArp and MinArp (and Int5 and Int7). However, its cost on scales (particularly ChrSc) is quite high compared to instruments 1.2, 7.1, and 8.

The preceding discussion has shown us which basic instruments are best for some common sequences of notes. But are the basic instruments good examples from the larger class of partitionable instruments, in which we are more interested? In the appendix, we provide evidence that this is the case by showing that optimal basic instruments are often optimal partitionable instruments, for both primary and free fingerings. A key exception is for the collections MajArp, MinArp, Int5, and Int7, where we show that one can do better with a non-basic partitionable instrument (Opt for Int5).

Let us point out that Proposition 1 in the appendix is generalized in two ways in Remark 8 in Section 4. This analysis is not complete and suggests some open problems (see Section 6).

4 Robustness and Experimentation

In this section we reconsider our choice of the function $Hdist$, which measures the difficulty of playing one note after another. We consider some alternative functions and check the robustness of our main results for the two instruments 1.2 and 1.7. We also propose an experiment that could be conducted to estimate this function.

As we mentioned in Section 2.2, the $Hdist$ function is one possible measure of the difficulty of playing two notes, one after the other. However, it may not be the case, for example, that moving two fingers is, on average, twice as difficult as moving one finger, or that moving three fingers is, on average, three times as difficult as moving one finger, as the $Hdist$ function implies. To generalize this notion of difficulty, let F be an increasing function from $\mathbb{N} = \{1, 2, 3, \ldots\}$ to $\mathbb{R}$. Then we define $Fdist(f_1, f_2)$, for fingering f_1 and f_2, as follows:

$$Fdist(f_1, f_2) = F(Hdist(f_1, f_2)).$$

We obtain the following general result for instruments 1.2 and 7.1.

Remark 8 *By slightly modifying the proof of Proposition* **??***, we can show the following stronger result. Consider any sequence of notes that spans at most one octave and has, between adjacent notes, h intervals of size 1, w intervals of size 2, and no intervals with other sizes. Suppose $\frac{h}{2} \leq w$. (Examples of such note sequences include the major, melodic minor (ascending and descending), acoustic major, octatonic, whole tone, and diminshed scales (as well as Int2). See [25] for definitions of these scales.) Furthermore, consider any $Fdist$ function (where, by definition, F is increasing). Then instrument 7.1 is an optimal* *partitionable instrument using primary fingerings where the $Fdist$ function is used in place of the $Hdist$ function (and we define the appropriate sequence collections). In addition, it is easy to check that instrument 1.2 remains optimal for any such $Fdist$ function on ChrSc.*

To further check the robustness of our results, consider the following examples of functions F:

$$
\begin{aligned}
F(x) &= x^{\frac{3}{2}} \\
F(x) &= x^{\frac{1}{2}} \\
F(x) &= 2x - 1 \\
F(x) &= \frac{1}{2}x + \frac{1}{2}
\end{aligned}
$$

These functions F have the following nice properties: they are increasing in x; they are normalized so that $F(1) = 1$; one example is strictly concave and one is strictly convex; two examples are linear; two examples have values higher than $F(x) = x$ and two have values lower than $F(x) = x$; and the range of these examples appears to be large enough to include all reasonable real difficulties (see discussion below).

Redoing some of the calculations from Section 3 using the $Fdist$ function based on the above choices for F, we find that instrument 7.1 remains an optimal instrument using primary fingerings for melodies and MinHSc (over basic instruments). However, instrument 7.1 remains optimal for only two of the alternate cost functions for pentatonic scales.

A related issue is the following: Can we experimentally determine the function F? A key factor in the notion of "difficulty" is being able to depress or release a set of keys as simultaneously as possible. On instruments such as woodwinds, and especially on very touch sensitive instruments such as the EWI, the more simultaneous the movements, the better the sound, especially when playing note sequences that are connected by slurs or are played rapidly.

One way to measure the difficulty would be to have a musician try to change from one specified fingering to another precisely at a tick of a metronome. Using sensors, we could measure the amount of time during which a wrong note (or notes) is being played and use this as a meaure of difficulty: the larger the error, the more difficult the change in fingerings. Let t be the time of the metronome tick for switching fingerings. Consider a pair of fingerings that differ on n keys. Let $t_1, \ldots, t_n$ be the actual times each finger switches positions. Let us assume $t_1 < \cdots < t_n$. If $t_1 < t < t_n$, then the error is $t_n - t_1$. If $t_n < t$, then the error is $t - t_1$. If $t < t_1$, then the error is $t_n - t$. The experiment could be performed as follows: For each value of n, we have each subject repeatedly perform a fingering change for a variety of different fingering changes and then average the errors. An experiment of this nature is reported in [29].

5 Conclusions

The basic instrument 7.1 is an optimal partitionable instrument using primary fingerings for playing collections of major, melodic minor (ascending and descending), whole tone, acoustic, octatonic, and diminished scales (as well as Int2 and Int10). Instrument 7.1 is also an optimal basic instrument for harmonic minor scales using both primary and free fingerings. Furthermore, instrument 7.1 is an optimal basic instrument for pentatonic scales using primary fingerings (and is nearly optimal for free fingerings) and is an optimal basic instrument for playing real-world melodies in the collections from European and folk music. (Instrument 7.1 retains these optimality properties for the alternate cost functions we considered, except for the case of pentatonic scales.) For playing chromatic scales, the basic instrument 1.2 is an optimal partitionable instrument (along with the Bleauregard) using both primary and free fingerings (and it retains this property for the alternate cost functions we considered). For playing major, minor, and major 7 arpeggios, instrument 9 is an optimal basic instrument using primary and free fingerings; and the instruments OptInt5 and OptInt7 are partititionable (non-basic) instruments that are even better using primary and free fingerings on major and minor arpeggios. For playing dominant 7 arpeggios, instrument 8 is an optimal basic instrument using primary and free fingerings. We found that instruments 8 and 9 are optimal partionable instruments for the diminished 7 arpeggios and the augmented triad arpeggios, respectively.We also found that the optimal partitionable instruments over all collections of intervals using primary fingerings are basic, with the exception of Int5 and Int7, whose optimal partitionable instruments are OptInt5 and OptInt7. Finally, the optimal partitionable instruments over the collections Int1, Int4, Int8, and Int12 using free fingerings are also basic.

We have observed that the total costs for the optimal partitionable instruments we have identified on seven keys (the minimum possible for the range of the instruments) cannot be improved for the associated optimal sequence collections by considering partitionable instruments with more keys.

Finally, we have performed a robustness check for the instruments 1.2 and 1.7 and we find that they remain optimal in a number of situations.

In conclusion it seems that the basic instruments provide a good source of instrument designs that have low total cost for playing common sequences of notes. There is no basic instrument that is optimal for all the sequences of notes we considered. However, given our results on melodies from Western music and the importance of scales in Western music (and given our robustness checks), one could argue that a real instrument based on instrument 7.1 (and implemented as in Remark 1) would perform well in practice.

6 Open questions

Although we have been able to find optimal partitional instruments for a number of situations, some questions remain unresolved. For example, is instrument 7.1

an optimal partitionable instrument for harmonic minor or pentatonic scales using primary or free fingerings? (These scales include intervals of size 3.) Is instrument 8 an optimal partitionable instrument for dominant 7 arpeggios using primary or free fingerings? Is instrument OptInt5 (equivalently OptIn7) an optimal partitionable instrument on MajArp and MinArp using primary or free fingerings? Perhaps a variation on the proof of Proposition **??**, part 1 could resolve some of these questions.

Our basic instruments are constructed from Gray code matrices of a particular common type with a particular type of row labels and primary fingerings. For cases where we do not know the optimal partitionable instruments, could basic instruments based on different Gray code matrices and/or different row labels and/or different primary fingerings yield lower cost instruments?

Here are some other questions to consider: Are there other note sequences that would be important to consider in searching for low cost instruments? Can a better model of difficulty of playing sequences of notes be obtained from experiments as outlined in Section 4? Are there other situations, such as the one discussed in Remark 1, where more than seven keys could be advantageous?

7 Appendix

In this appendix we present complete proofs of several results presented in this paper.

Calculation of number of different mappings for the primary fingerings on the keys $4, 5, 6, 7$: There are 16 different fingerings on the keys $4, 5, 6, 7$. Hence, there are $\binom{16}{12}$ different ways to choose 12 primary fingerings in a given octave and 12! is the number of different ways to order a particular choice of 12 fingerings, each of which corresponds to a different mapping from fingerings to pitch classes. Thus, we get $\binom{16}{12} \cdot 12! \approx 8.7 \times 10^{11}$ different mappings for the primary fingerings on the keys $4, 5, 6, 7$. ■

Proof of Proposition 4: We prove the result in a slightly more general setting: We let the range be the maximal set of consecutive notes the intrument can play using nonnegative rows of its interval matrices. Let $I(B)$ be a nonnegative interval-based instrument matrix, with

$$B = \{B_1 = B(s_1, t_1, v_1), \ldots, B_m = B(s_m, t_m, v_m)\}.$$

Assume $s_1 \geq \cdots \geq s_m$. The result is clearly true for $m = 1$ (in which case $s_1 = 1$). Assume inductively that it is true for instruments with $m - 1 \geq 1$ matrices. Since we can play all notes in the original range with $B_1, \ldots, B_m$, then we must be able to play all notes in the smaller range $N(0), \ldots, N(s_1 - 1)$ with $B_2, \ldots, B_m$, since none of these notes can involve choosing a row from B_1 with non-zero row label. So we can apply the inductive hypothesis to $B_2, \ldots, B_m$ to see that the greedy algorithm produces, at least, all the notes $N(0), \ldots, N(s_1 - 1)$ using the matrices $B_2, \ldots, B_m$. But now it should be clear that the greedy algorthm also works for $B_1, \ldots, B_m$ over the original range. ■

Proof of Proposition 7: Observe that by Remark 2, all basic instruments derive from interval matrices $B(s_i, t_i, v_i)$ such that $v_i \leq 0$. We begin the proof by considering the case the $v_i = 0$. Observe that it suffices to enumerate those instruments that can play the notes $N(1), \ldots, N(11)$. The reason is that, by definition, each instrument can play $N(0)$; and, if it can play $N(0), \ldots, N(11)$, then it can play the entire range since each instrument uses $B(12, 3)$.

Since each instrument has three keys dedicated to $B(12, 3)$, it suffices to consider the following cases for the remaining four keys.

Case 1: One matrix with 4 keys.

Case 2: One matrix with 3 keys and one matrix with 1 key.

Case 3: Two matrices, each with 2 keys.

Case 4: One matrix with 2 keys and two matrices, each with 1 key.

Case 5: Four matrices, each with 1 key.

Case 1: The only solution is to use $B(1, 4)$.

Case 2: The only matrices with 3 keys to use are $B(1, 3)$ and $B(2, 3)$ because, if we use $B(s, 3)$ with $s \geq 3$, then there is no matrix with 1 key we can add to get both $N(1)$ and $N(2)$.

Suppose we use $B(1, 3)$. Then we can add $B(s, 1)$, for $4 \leq s \leq 8$, and get all the notes.

If we use $B(s, 1)$, for $s \leq 3$, we cannot get $N(11)$. If we use $B(s, 1)$, for $s \geq 9$, we cannot get $N(8)$.

Suppose we use $B(2, 3)$. Then we must add $B(1, 1)$ in order to get $N(1)$.

Case 3: We consider two matrices of the form $B(s, 2)$. If we do not choose $B(1, 2)$, then we cannot get $N(1)$. If we add $B(1, 2)$ or $B(2, 2)$, we cannot get $N(11)$. If we add $B(s, 2)$, for $s \geq 5$, we cannot get $N(4)$. If we add $B(3, 2)$ or $B(4, 2)$, we can get all the notes.

Case 4: We must choose one matrix $B(s, 2)$. If we choose $B(s, 2)$, for $s \geq 5$, then we cannot get $N(1), N(2), N(3)$, and $N(4)$ with two matrices, each with 1 key.

Suppose we choose $B(1, 2)$. The only way to get $N(11)$ is to choose $B(s_1, 1)$ and $B(s_2, 1)$ with $s_1, s_2 \geq 4$. The only way to get $N(4)$ is to add $B(4, 1)$. To this we can add $B(s_2, 1)$, for $4 \leq s_2 \leq 8$. We cannot add $B(s_2, 1)$, for $s_2 \geq 9$, since we could not get $N(8)$.

Suppose we choose $B(2, 2)$. We must add $B(1, 1)$ to get $N(1)$. We cannot add $B(1, 1)$, $B(2, 1)$, or $B(3, 1)$, because we could not get $N(11)$. We cannot add $B(9, 1)$, $B(10, 1)$, or $B(11, 1)$, because we could not get $N(8)$. The remaining matrices $B(s, 1)$, for $4 \leq s \leq 8$, can be added.

Suppose we choose $B(3, 2)$. To get $N(1)$ and $N(2)$, we must add either $B(1, 1), B(1, 1)$ or $B(1, 1), B(2, 1)$. Both choices work.

Suppose we choose $B(4, 2)$. To get $N(1)$, $N(2)$, and $N(3)$, we must add $B(1, 1)$ and $B(2, 1)$.

Case 5: The solution must contain one $B(1, 1)$ to get $N(1)$. In order to get $N(2)$, the solution must contain an additional $B(1, 1)$ or $B(2, 1)$.

Case 5.1: Suppose we add $B(1, 1)$. To get $N(3)$, we must add $B(1, 1)$, $B(2, 1)$, or $B(3, 1)$.

Case 5.1.1: Suppose we add $B(1,1)$. We have to add $B(8,1)$ to get $N(11)$, but then we cannot get $N(4)$.

Case 5.1.2: Suppose we add $B(2,1)$. We have to add $B(7,1)$ to get $N(11)$, but then we cannot get $N(5)$.

Case 5.1.3: Suppose we add $B(3,1)$. We have to add $B(6,1)$ to get $N(11)$. This works.

Case 5.2: Suppose we add $B(2,1)$ (and no additional $B(1,1)$ is used). To get $N(4)$, we must add $B(2,1)$, $B(3,1)$, or $B(4,1)$.

Case 5.2.1: Suppose we add $B(2,1)$. We must add $B(6,2)$ to get $N(11)$. This works.

Case 5.2.2: Suppose we add $B(3,1)$. We must add $B(s,1)$, for $s \geq 5$ to get $N(11)$. If we add $B(s,1)$, for $s \geq 8$, we cannot get $N(7)$. Each of $B(5,1)$, $B(6,1)$, and $B(7,1)$ works.

Case 5.2.3: Suppose we add $B(4,1)$. We must add $B(s,1)$, , for $s \geq 4$ to get $N(11)$. If we add $B(s,1)$, for $s \geq 9$ we cannot get $N(8)$. Each of $B(4,1)$, $B(5,1)$, $B(6,1)$, $B(7,1)$,and $B(8,1)$ works.

To finish the proof it suffices to enumerate the instruments with $v_i < 0$ for at least one interval matrix. To see that Figure 7 contains the complete list of such instruments, consider an instrument $I(B)$ with an interval matrix $B(s_i,t_i,v_i)$ for B such that $v_i < 0$. Consider the instrument $I'(B)$ obtained by changing just this one interval matrix by adding s_i to each of its labels. It is easy to see that $I'(B)$ also satisfies our conditions. Hence $I(B)$ can be transformed to an instrument that satisfies our conditions with $v_i = 0$ for all its interval matrices. Hence any instrument that satisfies our conditions and has $v_i < 0$ for at least one of its interval matrices, can be obtained from one of the 32 basic instruments in Figure 7 by reducing various values of v_i by increments of s_i. Using this fact, the second list above was created by straightforward enumeration. ■

Proof of Proposition ??, part 1:

(The proof given here is similar to the proof of Theorem 3 in [13].) The extension to prove the Remark 8 is minor.

Observe that whenever a sequence crosses from one octave to the next higher octave, it costs 1 on keys $1,2,3$, for any choice of primary fingerings. Hence we focus our attention on the cost of playing these sequences on keys $4,5,6,7$. In particular, we let $Hdist'(pr(N(k),pr(N(l))$ equal the Hamming cost on keys $4,5,6,7$ of playing $N(k)$ then $N(l)$.

Let us divide up the intervals of size 1 and 2 in the range $N(0),\ldots,N(95)$ into the following classes. For $i = 0,\ldots,11$, let $Int1(i)$ equal the set of intervals of size 1 in the range that start on the notes of the form $N(12j+i)$, where $j \geq 0$. Hence, for example, $Int1(0) = \{\{N(0),N(1)\},\{N(12),N(13)\},\ldots,\{N(84),N(85)\}\}$ and $Int1(1) = \{\{N(1),N(2)\},\{N(13),N(14)\},\ldots,\{N(85),N(86)\}\}$. Similarly, for $i = 0,\ldots,11$, let $Int2(i)$ equal the set of intervals of size 2 in the range that start on the notes of the form $N(12j+i)$, where $j \geq 0$. Observe that $Int1(i)$ contains 8 intervals for $i = 0,\ldots,10$ and only 7 intervals for $i = 11$. Similarly, $Int2(i)$ contains 8 intervals for $i = 0,\ldots,9$ and only 7 intervals for $i = 10,11$. Observe that, because our instrument is partitionable, we have that the cost of playing any interval in $Int1(i)$ is the same on keys $4,5,6,7$ for any

$i = 0, \ldots, 11$. Similarly, the cost of playing any interval in $Int2(i)$ is the same for any $i = 0, \ldots, 11$.

Consider the first 12 sequences in MajSc (i.e., those with first note in the first octave) and consider the first two notes, that is, the first interval in each of these sequences. These 12 intervals contain exactly one interval in $Int2(i)$, for each $i = 0, \ldots, 11$. The same is true for the second 12 sequences in MajSc. Hence the contribution to the total cost (on keys $4, 5, 6, 7$) of playing the first interval in all the sequences in MajSc is $7 \cdot \sum_{i=0}^{11} Hdist'(pr(N(i)), pr(N(i+2))$. Similarly, the contribution to the total cost of playing the second interval in all the sequences in MajSc is $7 \cdot \sum_{i=0}^{11} Hdist'(pr(N(i)), pr(N(i+2))$. The contribution to total cost of playing the third interval in all the sequences in MajSc is $7 \cdot \sum_{i=0}^{11} Hdist'(pr(N(i)), pr(N(i+1))$, since this interval has size 1; and so on. Since there are 5 intervals of size 2 in each sequence in MajSc, the total cost of playing all the intervals of size 2 in MajSc is $5 \cdot 7 \cdot \sum_{i=0}^{11} Hdist'(pr(N(i)), pr(N(i+2))$; and, since there are 2 intervals of size 1 in each sequence in MajSc, the total cost of playing all the intervals of size 1 in MajSc is $2 \cdot 7 \cdot \sum_{i=0}^{11} Hdist'(pr(N(i)), pr(N(i+1))$.

Observe that

$$\sum_{i=0}^{11} Hdist'(pr(N(i)), pr(N(i+1)) = \sum_{i=0}^{11} Hdist'(pr(N(i+1)), pr(N(i+2)).$$

Hence, the total cost of playing all the sequences in MajSc can be written as follows

$$\sum_{i=0}^{11} \left\{ \begin{array}{c} 7 \cdot Hdist'(pr(N(i)), pr(N(i+1))+ \\ 7 \cdot Hdist'(pr(N(i+1)), pr(N(i+2))+ \\ 35 \cdot Hdist'(pr(N(i)), pr(N(i+2)) \end{array} \right\} \tag{1}$$

Observe that each bracketed term in (1) involves three parts based on an interval of size 2 and the two intervals of size 1 "contained in" that interval of size 2. It is easy to see that the best possible cost to play three such intervals involves having two at cost 1 and one at cost 2 (since it is not possible to have all three at cost 1). Furthermore, due to the coefficients of 7 and 35, it is optimal to have the cost of 2 on the part with the interval of size 1. This is precisely what instrument 7.1 achieves. ■

Proof of the remainder of Proposition ??: Consider Int4. For octave 1, consider the 4 sequences of notes: $N(0), N(4), N(8), N(12)$; $N(1), N(5), N(9), N(13)$; $N(2), N(6), N(10), N(14)$; $N(3), N(7), N(11), N(15)$. For octave 2 consider the 4 sequences of notes $N(12), N(16), N(20), N(24)$; $N(13), N(17), N(21), N(25)$; $N(14), N(18), N(22), N(26)$; $N(15), N(19), N(23), N(27)$. Consider the similar sequences for the next 5 octaves. Observe that these sequences have the property that the total cost of playing them has the same total cost as the collection Int4. The best possible cost of playing each sequence, on the non-octave keys, is 4, since if any two jumps have a cost of 1, then the third must have a cost of 2 (since its a different pitch class). The best possible cost of playing each sequence, on the octave keys, is 1, since each spans two octaves. Hence the best possible total cost for each sequence is 5. Since there are $4 * 7 = 28$

sequences, the best possible total cost for playing these sequences $5 * 28 = 140$. Instruments 9 and 14 achieve this cost.

Consider Int8. For octave 1, consider the 8 sequences of notes: $N(0), N(8), N(16), N(24)$; $N(1), N(9), N(17), N(25)$; and so on. For octave 3 consider the 8 sequences of notes $N(48), N(56), N(64), N(72)$; $N(49), N(57), N(65), N(73)$; and so on. Consider the similar sequences for octaves 5 and 7. Observe that these sequences have the property that the total cost of playing them has the same total cost as the collection Int8 plus the cost of playing the first 8 intervals of size 8 in octave 8. The best possible cost of playing each sequence, on the non-octave keys, is 4, since if any two jumps have a cost of 1, then the third must have a cost of 2 (since its a different pitch class). The best possible cost of playing each sequence, on the octave keys, is 2, since each spans three octaves. Hence the best possible total cost for each sequence is 6. Since there are $8*4 = 32$ sequences, the best possible total cost for playing these sequences $6 * 32 = 192$. Observe that 8 of these sequences each contain an interval that is outside of Int8. If each of these intervals contributes the maximum possible value of 3 to the total cost of the 32 sequences, then the best possible cost of playing Int8 is $192 - 8 * 3 = 168$. Instruments 9 and 14 achieve this cost. ■

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